ON EXTENDING MAPPINGS INTO NONLOCALLY
CONVEX LINEAR METRIC SPACES

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Abstract. It is proved that the following spaces are absolute retracts: every F-space
with a Schauder basis and certain function spaces along with their subgroups of
integer-valued elements. It is also observed that for every σ-compact convex set, the
absolute extension property for compacta implies the AR-property.

1. Introduction. The purpose of this paper is to provide new examples of
infinite-dimensional ANRs. Detecting the ANR-property of convex subsets of
nonlocally convex metric linear spaces and topological groups is of great importance.
For example, the topological classification of these spaces, given recently in [4, 5 and
3], required the ANR-property. We prove that the following spaces are absolute
retracts: (1) every complete metric linear space (= F-space) with a Schauder basis,
(2) certain function spaces which include $L_p \ (p \geq 0)$ and Orlicz spaces, and (3)
additive subgroups consisting of all integer-valued functions in certain function
spaces. Consequently, each of these spaces, when complete and separable, is homeo-
morphic to a Hilbert space [4, 5]. The argument used in verifying the AR-property of
the above examples is also employed to show that the AR and the AE(ε) (absolute
extension property for compacta) properties coincide for σ-compact convex sets.
This enables us to find a dense convex topological copy of $\Sigma$, the linear span of the
Hilbert cube in the Hilbert space $l_2$, in every separable infinite-dimensional com-
plete convex set.

Our approach is very elementary and mostly involves the natural equiconnected
structures of convex sets and contractible groups. We also employ the admissibility
notion introduced by Klee [9, 10].

2. Equiconnected spaces that are ANRs. A space $X$ is said to be locally equicon-
nected if there exists a map $c: V \times [0, 1] \to X$, where $V$ is a neighborhood of the
diagonal in $X \times X$, such that $c(x_1, x_2, 0) = x_1$, $c(x_1, x_2, 1) = x_2$, and $c(x, x, t) = x$
for every $x_1, x_2, x \in X$ and $t \in [0, 1]$. The map $c$ is called a local equiconnecting
function. The space $X$ is an equiconnected space, and $c$ is an equiconnnecting function
if $c$ is defined on $X \times X \times [0, 1]$. In the sequel we will be interested in the following
two examples of locally equiconnected spaces.
Example 1. A convex subset of a metric linear space is an equiconnected space with a naturally defined equiconnecting function \( c(x_1, x_2, t) = (1 - t)x_1 + tx_2 \).

Example 2. A (locally) contractible topological group is a (locally) equiconnected space. A (local) equiconnecting function can be defined by
\[
c(g_1, g_2, t) = g_2 \circ (h(e, t))^{-1} \circ h(g_1, t),
\]
where \( h \) is a homotopy with \( h(g, 0) = g \) and \( h(g, 1) = e \) for all \( g \) (a consequence: contractible topological groups are also locally contractible).

The following slight generalization of a theorem of Hanner [6] will be our main tool in applications.

**Theorem 1.** Let \( X \) be a locally equiconnected metric space. If \( \text{id}_X = \lim_{n \to \infty} (\beta_n \circ \alpha_n) \), where \( \alpha_n \) is a map of \( X \) into some \( X_n \in \text{ANR} \) and \( \beta_n \) is a map of \( X_n \) into \( X \), \( n = 1, 2, \ldots \), then \( X \in \text{ANR} \).

Theorem 1 can easily be derived from the following

**Lemma 1.** Let \( A \) be a closed subset of a metric space \((Z, \rho)\) and let \((X, d)\) be a locally equiconnected space. Suppose \( f: A \to X \) is a map such that \( f = \lim_{n \to \infty} f_n \), where each map \( f_n \) extends to a neighborhood of \( A \). Then \( f \) extends to a neighborhood of \( A \).

**Proof.** Let \( \{U_n\} \) be a sequence of open neighborhoods of \( A \) such that, for each \( n \), \( f_n \) extends to \( U_n \), \( \text{cl}(U_{n+1}) \subseteq U_n \), and \( \bigcap_{n=1}^\infty U_n = A \). For each \( z \in Z \) pick \( a_z \in A \) such that \( \rho(z, a_z) \leq 2\rho(z, A) \). We may assume each \( U_n \) is small enough so that for each \( z \in U_n \), \( d(f_n(z), f_n(a_z)) \leq 1/n \). Let \( \lambda: U_1 \setminus A \to (1, \infty) \) be a map such that for each \( n \), \( \lambda(U_n \setminus \text{cl}(U_{n+1})) \subseteq (n, n + 1) \) and \( \lambda(\text{bd}(U_{n+1})) = n + 1 \). Let \( c: V \to X \) be a local equiconnecting function defined on a neighborhood of the diagonal in \( X \times X \). For each \( n \geq 2 \), set \( V_n = \{z \in U_n : (f_{n-1}(z), f_n(z)) \in V\} \).

Consider \( W = \bigcup_{n=2}^\infty (V_n \setminus U_{n+1}) \cup A \). We claim that \( W \) is a neighborhood of \( A \) and \( f \) extends over \( W \). The hypothesis \( f = \lim_{n \to \infty} f_n \) means that, for each \( a \in A \) and \( \varepsilon > 0 \), there exists a neighborhood \( N(a) \) in \( A \) and an integer \( N \) such that for each \( a' \in N(a) \) and \( n \geq N \), \( d(f_n(a'), f_n(a)) < \varepsilon \). Together with the above condition on each \( U_n \), this implies the following:

(1) For each \( a \in A \) and \( \varepsilon > 0 \), there exists a neighborhood \( U(a) \) in \( Z \) and an integer \( N \) such that for each \( n \geq N \) and \( z \in U(a) \cap U_n \), \( d(f_n(z), f(a)) < \varepsilon \).

Thus for \( U(a) \) small enough, we have \( U(a) \cap U_n \subseteq V_n \) for all large \( n \), hence \( U(a) \cap U_n \subseteq W \). An extension \( \tilde{f}: W \to X \) is defined by the formula
\[
\tilde{f}(a) = \begin{cases} c(f_{n-1}(z), f_n(z), \lambda(z) - n), & z \in V_n \setminus U_{n+1}, \\ f(z), & z \in A. \end{cases}
\]

3. Applications.

**Theorem 2.** Every F-space \( E \) with a Schauder basis (more generally: every F-space admitting a sequence \( T_n: E \to E \) of finite rank continuous linear operators such that \( T_n x \to x \) for every \( x \in E \)) is homeomorphic to a Hilbert space.

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\( \text{id}_X = \lim_{n \to \infty} (\beta_n \circ \alpha_n) \) means that \( \text{id}_X \), the identity on \( X \), is the limit in the compact-open topology of the sequence \( \{\beta_n \circ \alpha_n\} \).

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PROOF. The Banach-Steinhaus Theorem yields $\lim_{n \to \infty} T_n = \text{id}_E$. Since $T_n(E)$ is isomorphic to a Euclidean space, it is an AR. Applying Theorem 1, we obtain $E \in \text{AR}$, which implies [4] that $E$ is homeomorphic to a Hilbert space. □

The next example requires a preliminary description. Let $(S, \Sigma, \mu)$ be a $\sigma$-finite measure space. A function space is a metric linear space $(X, \| \cdot \|)$ of $\mu$-equivalence classes of real $\Sigma$-measurable functions on $S$ satisfying

(i) if $|f(s)| \leq |g(s)|$ a.e. and $g \in X$, then $f \in X$ and $\|f\| \leq \|g\|$, 
(ii) for every $\{f_n\} \subset X, |f_n(s)| \leq |f(s)|$ a.e. for $n = 1, 2, \ldots$, with $0 \leq f \in X$, we have $\|f_n\| \to 0$ iff $f_n \to 0$ in measure on every set of $\mu$-finite measure (cf. [8]).

Function spaces which are Banach spaces were intensively investigated in a series of papers by Luxemburg and Zaanen (see, e.g., [12]). In our setting we permit function spaces to be nonlocally convex ($\| \cdot \|$ is not supposed to be homogeneous; we only require $(f, g) \to \|f - g\|$ to be a metric on $X$).

**Theorem 3.** Every separable function space $X$ is an absolute retract.

**Proof.** First, we construct a set $E = \bigcup E_n, E_n \in \Sigma$ for $n = 1, 2, \ldots$, with the properties:

1. $X_E \cdot f = f$ for every $f \in X$, 
2. $X_{E_n} \subseteq X$, and 
3. $\mu(E_n) < \infty$ for $n = 1, 2, \ldots$.

To this end pick a dense subset $\{f_n\}_n$ of $X$. Write $E = \bigcup_{n=1}^{\infty} \{s: f_n(s) \neq 0\}$. Given $f \in X$ and $\varepsilon > 0$, there exists $f_n$ with $\|f - f_n\| < \varepsilon$. Using condition (i) we estimate

$$
\|f - X_E \cdot f\| \leq \|f - f_n\| + \|f_n - X_{E_n} f\| \leq 2 \cdot \varepsilon;
$$

this shows (1). Now, noting that

$$
\{s: f_n(s) \neq 0\} = \bigcup_{k=1}^{\infty} \{s: |f_n(s)| \geq k^{-1}\},
$$

it is easy to determine $E_n (n = 1, 2, \ldots)$ in order to fulfill (2) and (3).

We apply Theorem 1, by letting $\alpha_n(f) = X_{E_n} \cdot \text{sup}(\inf(n, f), -n)$ and $X_n = \{f \in X: X_{E_n} \cdot f = f \text{ and } |f(s)| \leq n \text{ a.e.}\}$. Clearly, by (2) and (i), $\alpha_n(X) \subseteq X_n$. It is elementary to show that $|\alpha_n(f) - \alpha_n(g)(s)| \leq |f(s) - g(s)|$ a.e. and $|\alpha_{n+1}(f) - f(s)| \leq |\alpha_n(f) - f(s)|$ a.e. for every $f, g \in X$ and $n = 1, 2, \ldots$. This, in turn, guarantees the equiuniform continuity of $\{\alpha_n\}$ by (i), and $\|\alpha_n(f) - f\| \to 0$ by (ii) and (1). Now, the fact that $\lim_{n \to \infty} \alpha_n = \text{id}_X$ easily follows. Moreover, observe that $X_n$ is a convex subset of $X$ satisfying: $\|f_n - f_0\| \to 0$ iff $f_n \to f_0$ in measure iff $f_n \to f_0$ in $L_1(E_n, \mu)$ for $\{f_n\}_{n=0}^{\infty} \subseteq X_n$. Thus $X_n$ is affinely homeomorphic to a convex subset of $L_1$. Finally, the Borsuk-Dugundji Theorem (see [2, p. 58]) ensures $X_n \in \text{AR}$ and Theorem 1 is applicable. □

Note 1. The separability condition in Theorem 3 may be dropped, assuming, additionally, $X_S \subseteq X$.

Note 2. The same proof yields $\{f \in X: |f(s)| \leq |f_0(s)| \text{ a.e.}\} \in \text{AR}$, where $f_0$ is any $\Sigma$-measurable nonnegative function.
Consider the subset $G$ of $L_2(0,1)$ consisting of all integer-valued functions. $G$ occurs as a closed additive subgroup of $L_2(0,1)$ admitting no one-parameter subgroups. Answering a question of Wojtynski, we describe the topological nature of $G$.

**Theorem 4.** Let $X$ be a separable function space defined on a σ-finite measure space $(S, \Sigma, \mu)$ with atomless measure $\mu$. The additive subgroup $G$ of $X$ consisting of all integer-valued functions is an AR. Moreover, if $G$ is complete, then $G$ is homeomorphic to $l_2$.

**Proof.** We will follow the proof of Theorem 3. Having constructed sets $E$ and $E_n$ ($n = 1, 2, \ldots$), we consider $\alpha_n(f) = \chi_{E_n} \cdot \text{sup}(\inf(n, f), -n)$ for $f \in G$. Clearly $\alpha_n(G) \subset \{ f \in G: \chi_{E_n} \cdot f = f \text{ and } |f(s)| \leq n \text{ a.e.} \} = G_n$. The argument used in Theorem 3 shows that $\lim_{n \to \infty} \alpha_n = \text{id}_G$, and the topology on $G_n$ coincides with the topology of convergence in measure. Hence, by a theorem of Bessaga and Pełczyński (see [2, p. 201]) the set $G_n$ is homeomorphic to $l_2$; so $G_n \in \text{AR}$.

Now, in order to apply Theorem 1 we will show that $G$ is contractible (cf. Example 2). Since $\mu$ is σ-finite and atomless, there exists a continuous map $\lambda: [0,1] \to (\Sigma, d)$, where $d(E_1, E_2) = \mu(E_2 \setminus E_1) + \mu(E_1 \setminus E_2)$, such that $\lambda(1) = \emptyset$, $\lambda(0) = S$ and $\mu(\bigcup_{\epsilon \geq \epsilon_0} \lambda(\epsilon)) < \infty$ for every $\epsilon_0 > 0$. Write $h(f, t) = \chi_{\lambda(t)} \cdot f$ and note that $h(f, 0) = f$ and $h(f, 1) = 0$ for every $f \in G$. If $||f_n - f|| \to 0$ and $t_n \to t$, then by (i),

$$||h(f_n, t_n) - h(f, t)|| \leq ||\chi_{\lambda(t_n)} \cdot f_n - \chi_{\lambda(t_n)} \cdot f|| + ||\chi_{\lambda(t_n)} \cdot f - \chi_{\lambda(t)} \cdot f||$$

Since $\chi_{\lambda(t_n)} \cdot f - \chi_{\lambda(t)} \cdot f \to 0$ in measure, the continuity of $h$ follows from (ii). The last part of Theorem 4 follows from [5].

### 4. Admissibility and AR properties

Assume $C$ is a convex subset of a metric linear space $(E, || \cdot ||)$. Following Klee [9], we say that $C$ is admissible if for every compact set $K \subset C$ the identity map $\text{id}_K$ can be uniformly approximated by maps $\phi: K \to C$ with $\dim \text{span}(\phi(K)) < \infty$. Klee [9] proved that admissible $F$-spaces are AE($\mathcal{F}$). The following theorem extends the result of Klee.

**Theorem 5.** A convex set $C$ is an AR iff $\text{id}_C = \lim_{n \to \infty} \phi_n$, where each $\phi_n$ is a locally finite-dimensional (l.f.d.) map, i.e., every point of $C$ has a neighborhood $U$ with $\dim \text{span}(\phi_n(U)) < \infty$.

We will employ the following

**Lemma 2.** Let $\{ A_n \}$ be a countable closed cover of a metric space $A$ and $C$ a convex subset of a metric linear space. Suppose $f: A \to C$ is a map such that for each $n$, $f|A_n$ is the uniform limit of l.f.d. maps. Then $f$ is the uniform limit of l.f.d. maps.

**Proof of Lemma 2.** Fix a map $f: A \to C$ and $\epsilon > 0$. Choose an l.f.d. map $f_n: A_n \to C$ with $||f_n - f|A_n|| < 2^{-n}\epsilon$. Since finite-dimensional convex subsets of $C$ are ARs, applying a partition of unity argument, we construct l.f.d. maps $f_n': A \to C$...
such that \( \tilde{f}\big| A_n = f_n \). There exists a locally finite partition of unity \( \{ \lambda_n \} \) with 
\( \lambda_n^{-1}\{(0,1]\} \subset \{ a \in A: \| f_n(a) - f(a) \| < 2^{-n}\varepsilon \} \). Letting 
\( \tilde{f}(a) = \sum_1^\infty \lambda_n(a) \tilde{f}_n(a) \), we see that \( \tilde{f} \) is an l.f.d. map satisfying 
\[
\| f(a) - \tilde{f}(a) \| \leq \sum_1^\infty \| \lambda_n(a) \| \| f(a) - \tilde{f}_n(a) \| \leq \sum_1^\infty 2^{-n}\varepsilon = \varepsilon. \]

**Proof of Theorem 5. Necessity.** Since \( C \in AR \), for every \( \varepsilon > 0 \) there exists a locally finite-dimensional metric simplicial complex \( K \) and maps \( \phi: C \to |K|, \psi: |K| \to C \) such that \( \| \psi \phi(x) - x \| < \varepsilon \). Clearly \( |K| \) is a countable union of closed finite-dimensional sets. It can be easily verified that every map of a finite-dimensional metric space into \( C \) is the uniform limit of l.f.d. maps (cf. [4]). Thus, by Lemma 2, the map \( \psi \) is the uniform limit of l.f.d. maps. Consequently, \( id_C \) is also the uniform limit of l.f.d. maps.

**Sufficiency.** Let \( f: A \to C \) be a map, where \( A \) is a closed subset of a metric space \( Z \). Using a partition of unity argument and the fact that finite-dimensional convex subsets of \( C \) are ARs, we construct maps \( f_n: Z \to C \) such that \( f_n|A = \phi_n \circ f \) for \( n = 1, 2, \ldots \). Since \( \lim_{n \to \infty} f_n = f \), Lemma 1 ensures that \( f \) extends to a neighborhood of \( A \). Finally, since \( C \) is a contractible ANR, it is an AR. \( \square \)

**Note 3.** The same proof (for the sufficiency part) yields \( C \in AE(\mathcal{F}) \) iff \( C \) is admissible.

**Note 4.** Applying Lemma 2 in the sufficiency part of the proof of Theorem 5, one can show that \( C \in AR \), provided that \( id_C = \lim_{n} \phi_n \) such that each \( \phi_n(C) \) is a countable union of closed finite-dimensional sets (cf. [1]). More generally, one may only require that each \( \phi_n(C) \) is a C-space (see [7]). This follows from the fact that Lemma 2 holds when \( A \) is a subspace of \( C \) with the weaker assumption that each \( A_n \) is a (not necessarily closed) C-space.

**Corollary 1.** For a \( \sigma \)-compact convex set the following conditions are equivalent:

(a) \( C \in AR \),

(b) \( C = \bigcup_1^\infty A_n \) with \( A_n = \text{cl}(A_n) \in AE(\mathcal{F}) \) for \( n = 1, 2, \ldots \).

(c) \( C \) is admissible.

**Proof.** This is a consequence of Theorem 5 and Lemma 2. \( \square \)

**Corollary 2.** Every separable infinite-dimensional complete convex set \( C \) contains a dense convex subset \( \tilde{C} \) homeomorphic to \( \Sigma = \{(x_n) \in l_2: \Sigma_1^\infty (nx_n)^2 < \infty \} \).

**Proof.** The proof of [3, Proposition 3.4] shows that there are affine copies \( C_1, C_2, \ldots \) of compact convex subsets of \( l_2 \) with \( \tilde{C} = \bigcup_1^\infty C_n \) dense in \( C \) and such that \( \{ C_n \} \) is a so-called strong universal tower for compacta. By Corollary 1 we have \( \tilde{C} \in AR \). Finally, the assertion follows from [3, Theorem 4.1]. \( \square \)

**5. A generalization.** The problem of whether a separable metric space \( X \in ANE(\mathcal{F}) \) is an ANR, posed by Kuratowski [11], remains unsolved. Apparently, an answer to this problem is not known even for \( \sigma \)-compact \( X \). Our consideration leads to a partial answer.
Proposition. Let \((X, d)\) be a \(\sigma\)-compact space and \(c\) a local equiconnecting function defined for all \(x_1, x_2\) with \(d(x_1, x_2) < \varepsilon\) for some \(\varepsilon > 0\) and satisfying

\[(*) \quad d(c(x_1, x_2, t), x_2) < d(x_1, x_2) \quad \text{for every } x_1, x_2 \in X \text{ and } 0 < t < 1.\]

If \(X = \bigcup_i^\infty X_i\) and each \(X_i = \text{cl}(X_i) \in \text{ANE}(\mathscr{C})\), then \(X \in \text{ANR}\).

Proof (Sketch). First, to show that \(X \in \text{ANE}(\mathscr{C})\), consider a map \(f: A \rightarrow X\), where \(A\) is a closed subset of a compact set \(Z\). Imitating the argument of Lemma 2 with \(A_n = f^{-1}(X_n)\), one can show that \(f\) is a uniform limit of maps defined on neighborhoods of \(A\) in \(Z\). Lemma 1 guarantees a neighborhood extension of \(f\). To repeat the same argument in the case where \(A\) is a closed \(\sigma\)-compact subset of a metric space, we use the fact that any map of a compact subset \(B\) of a metric space \(Y\) into an \(\text{ANE}(\mathscr{C})\) extends to a neighborhood of \(B\) in \(Y\). \(\square\)

Note 5. If \(X\) is a convex subset of a metric linear space \((E, || \cdot ||)\), then the equiconnecting function of Example 1 satisfies \((*)\) with respect to the metric \(d\) induced by \(|| \cdot ||\). If \(X = (G, d)\) is a locally contractible topological group and \(d\) is left-invariant, then the local equiconnecting function of Example 2 satisfies \((*)\), provided that

\[(**)^{(*)} \quad d(h(g, t), h(e, t)) < d(g, e) \quad \text{for every } g \in G \text{ and } 0 < t < 1.\]

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