A WEIGHTED WEAK TYPE INEQUALITY FOR THE MAXIMAL FUNCTION

E. SAWYER

Abstract. We show that the operator $S = v^{-1}Mv$, where $M$ denotes the Hardy-Littlewood maximal operator, is of weak type $(1,1)$ with respect to the measure $v(x)w(x) \, dx$ whenever $v$ and $w$ are $A_1$ weights. B. Muckenhoupt's weighted norm inequality for the maximal function can then be obtained directly from the P. Jones factorization of $A_p$ weights using interpolation with change of measure.

In [8], B. Muckenhoupt characterized the nonnegative functions, or weights, $w$ on $\mathbb{R}$ satisfying the weighted norm inequality ($1 < p < \infty$)

$$\int_{-\infty}^{\infty} |Mf(x)|^p w(x) \, dx \leq C \int_{-\infty}^{\infty} |f(x)|^p w(x) \, dx$$

for all $f$,

where $Mf(x) = (\sup_{y \in I} 1/|I| |f(y)| \, dy$ is the Hardy-Littlewood maximal function of $f$, as those weights $w$ satisfying the $A_p$ condition

$$(A_p) \quad \left[ \frac{1}{|I|} \int_I w \right] \left[ \frac{1}{|I|} \int_I w^{1-p'} \right]^{p-1} \leq C \quad \text{for all intervals } I.$$

Later, P. Jones showed in [7] that a weight $w$ satisfies the $A_p$ condition if and only if it admits a factorization

$$w = w_0 w_1^{-p} \text{ where } Mw_j(x) \leq Cw_j(x) \text{ for all } x, j = 0, 1.$$

More recently, M. Christ and R. Fefferman [2] have given an elementary proof of the implication $(A_p) \Rightarrow (1)$ (see also R. Hunt, D. Kurtz and C. Neugebauer [5], B. Jawerth [6] and E. Sawyer [10]), and R. Coifman, P. Jones and J. Rubio de Francia [4] have given a short proof that (2) follows from (1) and its "dual" inequality, the boundedness of $M$ on $L^{p'}(w^{1-p'})$.

A natural approach to obtaining inequality (1) directly from the factorization in (2) is provided by the Stein-Weiss interpolation with change of measures theorem [11]. In order to see what is needed, suppose (2) holds and, following the proof in [11], define $Sf = w_1^{-1}M(w_1 f)$. Note that (1) can be rewritten

$$\int |Sf|^p w_0 w_1 \leq C \int |f|^p w_0 w_1 \quad \text{for all } f.$$
Now $S$ is bounded on $L^\infty(w_0w_1)$ simply because $Mw_1 \leq Cw_1$ and thus (3) will follow from the usual Marcinkiewicz interpolation theorem provided that $S$ is of weak type $(1,1)$ with respect to the measure $w_0(x)w_1(x) \, dx$.

**Theorem.** Suppose $v$ and $w$ satisfy the $A_1$ condition, i.e. $Mv(x) \leq Av(x)$ and $Mw(x) \leq Bw(x)$ for all $x$. Then

$$\int_{\{Mg \geq v\}} v(x)w(x) \, dx \leq C \int_{-\infty}^{\infty} g(x)w(x) \, dx \quad \text{for all } g \geq 0 \text{ on } R,$$

where the constant $C$ depends only on $A$ and $B$. This shows (with $g = \lambda^{-1}f\nu$) that the operator $Sf = \nu^{-1}M(vf)$ is of weak type $(1,1)$ with respect to $v(x)w(x) \, dx$.

It would be of interest to obtain an analogue of the above approach for two weight inequalities and other operators, specifically the Hilbert transform. Regarding earlier work and other weighted weak type inequalities for the maximal function, see K. Andersen and B. Muckenhoupt [1], B. Muckenhoupt [8] and especially the treatment given by B. Muckenhoupt and R. L. Wheeden in [9] from which our proof borrows heavily. The letter $C$ will denote a positive constant that may change from line to line and $|E|_v = \int_E v(x) \, dx$, $|E|_w = \int_E w(x) \, dx$ for $v \geq 0$ on $R$, $E \subseteq R$.

**Proof.** It suffices to prove (4) for $g \geq 0$ bounded with compact support. Fix such a $g$. For $k \in \mathbb{Z}$, let \( \{I_j^k\} \) be the collection of component intervals of the open set \( \Omega_k = \{Mv > 3^k\} \cap \{Mg > 3^k\} \). Denote by $\Gamma$ the set of pairs $(k, j)$ such that $I_j^k \cap \{v < 3^{k+1}\}$ has positive measure. For $(k, j) \in \Gamma$ we then have

$$\frac{3^k}{A} \leq A^{-1} \operatorname{ess inf}_{I_j^k} Mv \leq \operatorname{ess inf}_{I_j^k} v \leq \left|I_j^k\right|^{-1} \int_{I_j^k} v$$

$$\leq \operatorname{ess inf}_{I_j^k} Mv \leq A \operatorname{ess inf}_{I_j^k} v \leq 3^{k+1}A,$$

since $Mv \leq Av$. Thus

$$\int_{\{Mg \geq v\}} vw \leq 3 \sum_{k} 3^k \left(\{3^k < v \leq 3^{k+1}\} \cap \{Mg > v\}\right)_w$$

$$\leq 3 \sum_{k} 3^k \sum_{j: (k, j) \in \Gamma} \left|I_j^k\right|_w$$

$$\leq 3A \sum_{(k, j) \in \Gamma} \left|I_j^k\right|^{-1} \left|I_j^k\right|_w \left|I_j^k\right|_w \text{ by (5)}.$$

For $N \in \mathbb{Z}$, set $\Gamma_N = \{(k, j) \in \Gamma: k \geq N\}$. We shall prove

$$\sum_{(k, j) \in \Gamma_N} \left|I_j^k\right|^{-1} \left|I_j^k\right|_v \left|I_j^k\right|_w \leq C \int g w \quad \text{for all } N$$

with a constant $C$ independent of $N$. The proof uses a variant of an idea of B. Muckenhoupt and R. L. Wheeden [9; Proof of Lemma 2]. First note that for $(k, j)$, $(t, s) \in \Gamma_N$ with $k \geq t$, either $I_j^k \subset I_s^t$ or $I_j^k \cap I_s^t = \emptyset$. Fix $N$ and let $G_0$ consist of the indices $(k, j) \in \Gamma_N$ for which $I_j^k$ is maximal in $\{I_j^k: (t, s) \in \Gamma_N\}$. Since $Mv \leq Av$, $v$ satisfies the $A_\infty$ condition [3] and thus there are positive constants $C$, $\varepsilon$ such that

$$|E|_v/|I|_v \leq C(|E|/|I|)^{\varepsilon} \quad \text{whenever } E \text{ is a subset of an interval } I.$$

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Choose $0 < \delta < \varepsilon$. If $G_n$ has been defined, let $G_{n+1}$ consist of those $(k, j) \in \Gamma$ for which there is $(t, s) \in G_n$ with $I^k_t \subset I'_s$ and

\[ (i) \quad \frac{1}{|I^k_t|} \int_{I^k_t} w > 3^{(k-t)\delta} \frac{1}{|I'_s|} \int_{I'_s} w, \]

(9)

\[ (ii) \quad \frac{1}{|I'_s|} \int_{I'_s} w \leq 3^{(t-s)\delta} \frac{1}{|I'_s|} \int_{I'_s} w \quad \text{whenever } (t, s) \in \Gamma \text{ and } I^k_t \subset I'_s \subset I'_s. \]

Let $P = \bigcup_{n=0}^{\infty} G_n$. Following [9], we claim that

\[ (10) \quad \sum_{(k, j) \in \Gamma_n} |I^k_t|^{-1}|I'_s|^{-1}|I^k_t|_{w} - C \sum_{(k, j) \in P} |I^k_t|^{-1}|I'_s|^{-1}|I^k_t|_{w}. \]

To see this, suppose $(t, s) \in P$ and let $Q = Q(t, s)$ denote the set of indices $(k, j) \in \Gamma$ such that $I^k_t \subset I'_s$ and there is no $(l, i) \in P$ with $I^k_l \subset I'_i \subset I'_s$. Then by (9)(ii)

\[ \sum_{(k, j) \in Q} |I^k_t|^{-1}|I'_s|^{-1}|I^k_t|_{w} \leq \sum_{(k, j) \in Q} 3^{(k-t)\delta}|I^k_t|^{-1}|I'_s|^{-1}|I^k_t|_{w}. \]

\[ \leq |I^k_t|^{-1}|I'_s|^{-1} \sum_{k=t}^{\infty} 3^{(k-t)\delta}|I^k_t|_{w} C \left( \frac{|\{M^w > 3^k\} \cap I'_s|_w}{|I'_s|} \right)^\varepsilon \quad \text{by (8)}. \]

However, $M^w \leq A^w$ and so $|\{M^w > 3^k\} \cap I'_s|_w \leq A^w 3^{-k}|I'_s|_w \leq A^2 3^{(t-k+1)}|I'_s|_w$ by (5) and thus the final line above is dominated by

\[ CA^2 |I^k_t|^{-1}|I'_s|^{-1}|I^k_t|_{w} \sum_{k=t}^{\infty} 3^{(k-t)\delta} 3^{(t-k+1)\varepsilon} \]

\[ \leq C |I^k_t|^{-1}|I'_s|^{-1}|I^k_t|_{w} \quad \text{since } 0 < \delta < \varepsilon. \]

Inequality (10) now follows since $\bigcup_{(t, s) \in P} Q(t, s) = \Gamma_N$.

For each $k$ in $\mathbb{Z}$, let $\{J^k_t\}_i$ be the component intervals of $\{M^w > 3^k\}$. Then $M(\chi_{J^k_t} g) > 3^k$ on $J^k_t$ and so

\[ (11) \quad |J^k_t| \leq \left| \left\{ M(\chi_{J^k_t} g) > 3^k \right\} \right| \leq C 3^{-k} \int_{J^k_t} g, \]

since the maximal operator is of weak type $(1, 1)$ with respect to Lebesgue measure. Given an interval $I^k_t$, there is a unique $i = i(k, j)$ such that $I^k_t \subset J^k_i$. From now on, whenever the index $i$ appears in a summation over $(k, j)$, it is understood that $i = i(k, j)$. We have

\[ (12) \quad \sum_{(k, j) \in P} |I^k_t|^{-1}|I'_s|^{-1}|I^k_t|_{w} \leq 3A \sum_{(k, j) \in P} 3^k|I^k_t|_{w} \quad \text{by (5)} \]

\[ \leq CA \sum_{(k, j) \in P} \frac{1}{|I^k_t|} \left( \int_{J^k_t} g \right)|I^k_t|_{w} \quad \text{by (11)} \]

\[ = CA \int \left[ \sum_{(k, j) \in P} |J^k_i|^{-1}|I^k_t|_{w} \chi_{J^k_i} \right] g. \]
Let \( h(x) = \sum_{(k, j) \in P} \left| J_k \right|^{-1} \left| I_k^j \right| w_{X^{J_k}}(x) \). It remains to show that \( h(x) \leq Cw(x) \) for all \( x \in R \). So fix \( x \in R \). For any given \( k \), there is at most one interval \( J_k^j \) containing \( x \). We denote this interval, when it exists, by \( J_k^j \). Let \( P_k = \{(k, j) \in P: J_k^j \subset J_k^j\} \) and let \( G = \{k: P_k \neq \emptyset\} \). Let \( k_0 \) be the least integer \( k \) in \( G \) and if \( k_0, k_1, \ldots, k_n \) have been defined, choose \( k_{n+1} \) in \( G \) such that \( k_{n+1} > k_n \) and

\[
(i) \quad \frac{1}{\left| J_{k_{n+1}} \right|} \int_{J_{k_{n+1}}} w \geq 2 \frac{1}{\left| J_{k_n} \right|} \int_{J_{k_n}} w,
\]

\[
(ii) \quad \frac{1}{\left| J_l \right|} \int_{J_l} w \leq 2 \frac{1}{\left| J_{k_n} \right|} \int_{J_{k_n}} w \quad \text{for } k_n \leq l < k_{n+1}, l \in G.
\]

We now claim that

\[
\sum_{l \in G} \sum_{k_n \leq l < k_{n+1}} \frac{|I_l^j|}{|J_l^j|} w \leq C \quad \text{for } n > 0.
\]

First, we observe that if \( (l, j) \in P, k_n \leq l < k_{n+1}, \) then

\[
\frac{1}{|I_l^j|} \int_{I_l^j} w > \frac{3(l-k_n)^{\delta}}{2B} \frac{1}{|J_l^j|} \int_{J_l^j} w.
\]

To see this, let \( I_{l, s}^k \) be the component of \( \Omega_{k_n} \) that contains \( I_l^j \). We claim that \( (k_n, s) \in \Gamma \). Since \( P \subset \Gamma \), it suffices to consider the case \( (k_n, s) \notin P \). Since \( k_n \in G \), \( J_{k_n} \) must contain at least one interval of the form \( I_{u, s}^k \) with \( (k_n, u) \in P \) and it follows that \( J_{k_n} \supseteq I_{s, s}^k \). By the definition of \( \Omega_{k_n} \), we must have \( Mv \leq 3^{k_n} \) at one of the endpoints of \( I_{s, s}^k \). Thus the average of \( v \) over \( I_{s, s}^k \) is at most \( 3^{k_n} \) and so

\[
|\{ v \leq 3^{k_n+1} \} \cap I_{s, s}^k | > 0.
\]

Hence \( (k_n, s) \in \Gamma \) by definition. Now let \( I_s^j \) denote the smallest interval containing \( I_{s, s}^k \) with \( (k_n, s) \in P \). Sufficiently many applications of (9)(i) yield

\[
\frac{1}{|I_s^j|} \int_{I_s^j} w > \frac{3(k_n-k)^{\delta}}{2B} \frac{1}{|J_s^j|} \int_{J_s^j} w
\]

and, since \( (k_n, s) \in \Gamma \), (9)(ii) shows that

\[
\frac{1}{|I_{s, s}^k|} \int_{I_{s, s}^k} w \leq \frac{3(k_n-k)^{\delta}}{2B} \frac{1}{|I_{s, s}^k|} \int_{I_{s, s}^k} w.
\]

Finally, from (13)(ii) we have

\[
\frac{1}{|J_l^j|} \int_{J_l^j} w \leq \frac{2}{|J_{k_n}^j|} \int_{J_{k_n}^j} w \leq 2 \text{ess inf} w \leq \frac{2B}{|I_{s, s}^k|} \int_{I_{s, s}^k} w
\]

and, combining this with the two previous inequalities, we obtain (15). From (15) and the assumption \( Mw \leq Bw \) we obtain that for \( k_n \leq l < k_{n+1} \)

\[
\bigcup_{j: (l, j) \in P} I_l^j \subset \left\{ w > \frac{3(l-k_n)^{\delta}}{2B^2 |J_l^j|} \right\} \cap J_l^j.
\]
Since \( w \) satisfies the \( A_\infty \) condition [3], there are positive constants \( C, \eta \) such that 
\[
|E|_w/|I|_w \leq C(|E|/|I|^\eta)
\]
whenever \( E \subset \) an interval \( I \). Taking for \( E \) the set on the right side of (16), we conclude from the \( A_\infty \) condition and the inequality 
\[
|\{ w > \lambda \} \cap \lambda^{-1}|J|_w \|
\]
that \( |E|_w \) is dominated by \( |J|_w \) times \( C(2B^2/3^{(\frac{1}{r} + \delta)})^\eta \). It follows that the left side of (14) is dominated by \( \sum_{\lambda < \delta} C(3^{(\frac{1}{r} + \delta)})^\eta \leq C \), as required.

We can now complete the proof. We have
\[
h(x) = \sum_{(k, j) \in \mathcal{P}} \frac{|J_k^j|_w}{|J_k^j|_w} \left( \frac{1}{|J_k^j|_w} \int_{J_k^j} w \right) X_{J_k^j}(x)
\]
\[
= \sum_n \sum_{J \in \mathcal{G}} \sum_{k_n < j < k_{n+1}} \left( \sum_{(l, j) \in \mathcal{P}, |J_l^j|_w} \frac{|J_l^j|_w}{|J_l^j|_w} \left( \frac{1}{|J_l^j|_w} \int_{J_l^j} w \right) \right)
\]
\[
\leq \sum_n 2C \left( \frac{1}{|J_n^{k_n}|_w} \int_{J_n^{k_n}} w \right)
\]
by (13)(ii) and (14). By (13)(i), this last sum is dominated by twice its largest term which in turn is dominated by \( CMw(x) \leq CBw(x) \). Thus \( h(x) \leq Cw(x) \) and, combining this with (10) and (12), we obtain (7). Letting \( N \downarrow -\infty \) and using (6), we obtain (4).

REFERENCES