CRITERIA FOR A BLASCHKE QUOTIENT TO BE OF UNIFORMLY BOUNDED CHARACTERISTIC

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Abstract. Criteria for a quotient $B_1/B_2$ of Blaschke products $B_1$ and $B_2$ to be of uniformly bounded characteristic are proposed in terms of interpolating sequences.

A Blaschke product is a holomorphic function

$$B(z; \{a_n\}) = \prod_{n=1}^{\infty} \left( \frac{|a_n|}{a_n} \right) \frac{a_n - z}{1 - \bar{a}_nz}$$

in the disk $D = \{ |z| < 1 \}$, where $\{a_n\}$ is a sequence of complex numbers in $D$ with $\Sigma(1 - |a_n|) < \infty$, with the convention $|a_n|/a_n = 1$ for $a_n = 0$. A Blaschke quotient is a meromorphic function $B_1/B_2$, where $B_1$ and $B_2$ are Blaschke products with no common zero. J. A. Cima and P. Colwell [2, Theorem 2] established a criterion for $B_1/B_2$ to be normal in $D$ in the sense of O. Lehto and K. I. Virtanen [4] in terms of interpolating sequence in the sense of L. Carleson [1]. Here a function $f$ meromorphic in $D$ is normal if $(1 - |z|^2)f^*(z)$ is bounded where, $f^* = |f|/(1 + |f|^2)$, and a sequence of points $\{z_n\}$ in $D$ is interpolating (or uniformly separated [3, p. 148]) in $D$ if

$$\inf_{n \geq 1} \prod_{k=1, k \neq n}^{\infty} \left| \frac{z_k - z_n}{1 - \bar{z}_kz_n} \right| > 0.$$ 

If $\{z_n\}$ is interpolating in $D$, then $\Sigma(1 - |z_n|) < \infty$. Cima and Colwell's cited result is (I) $\iff$ (II) in

**Theorem.** Let $\{a_n^{(1)}\}$ and $\{a_n^{(2)}\}$ be disjoint interpolating sequences of points in $D$, and set

$$B_k(z) = B(z; \{a_n^{(k)}\}), \quad k = 1, 2.$$ 

Then the following are mutually equivalent.

(I) $B_1/B_2$ is normal in $D$.

(II) The sequence $\{a_n^{(1)}\} \cup \{a_n^{(2)}\}$ is interpolating in $D$.

(III) $B_1/B_2$ is of uniformly bounded characteristic in $D$.

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To explain the terminology we set \( f_w(z) = f((z + w)/(1 + wz)) \), \( z, w \in D \), for \( f \) meromorphic in \( D \). We call \( f \), of uniformly bounded characteristic in \( D \), \( f \in \text{UBC} \) for short, if
\[
T(1, f_w) = \lim_{r \to 1^-} T(r, f_w), \quad w \in D,
\]
is bounded in \( D \), where
\[
T(r, f_w) = \int_0^1 (\pi t)^{-1} \left[ \int_{|z| < t} (f_w)\pi(z)^2 \, dx \, dy \right] \, dt
\]
is the Shimizu-Ahlfors characteristic function of \( f_w \); see [5]. Since \( f = f_0 \), each \( f \in \text{UBC} \) is of bounded (Nevanlinna) characteristic in \( D \).

Since each \( f \in \text{UBC} \) is normal in \( D \) by [5, Theorem 3.1, p. 383], (III) \( \Rightarrow \) (I) is obvious. To establish our Theorem the remaining work is

**Proof of (II) \( \Rightarrow \) (III).** First we observe, for \( f = B_1/B_2 \), the identity
\[
(1) \quad T(1, f_w) = F^+(w) - F(w), \quad w \in D,
\]
where \( F = \frac{1}{2} \log(|B_1|^2 + |B_2|^2) \) and \( F^+ \) is the least harmonic majorant of the subharmonic function \( F \) in \( D \). For the proof of (1) we fix \( w \in D \), and we let
\[
D(w, r) = \{ f; |(f - w)/(1 - w\xi)| < r \}, \quad 0 < r < 1.
\]
Then the Green function of \( D(w, r) \) with its pole at \( w \) is
\[
g_r(\xi) = \log|r(1 - w\xi)/(\xi - w)|, \quad \xi \in D(w, r).
\]
Since the Laplacian \( \Delta F = 2f^\#_2 \) in \( D \), the change of variable \( z = (\xi - w)/(1 - w\xi) \), \( \xi \in D(w, r) \), together with
\[
T(r, f_w) = \pi^{-1} \int_{|z| < r} (f_w)\pi(z)^2 \log|r/z| \, dx \, dy,
\]
derived from [5, (2.5), p. 352] for \( f_w \), yields
\[
T(r, f_w) = (2\pi)^{-1} \int_{D(w, r)} (\Delta F(\xi)) g_r(\xi) \, d\xi \, d\eta.
\]
The Green formula
\[
\int_G (\phi \Delta \psi - \psi \Delta \phi) \, dx \, dy = \int_{\partial G} \left( \psi \frac{\partial \phi}{\partial \nu} - \phi \frac{\partial \psi}{\partial \nu} \right) \, ds
\]
for \( G = D(w, r) \setminus D(w, \epsilon) \) (\( 0 < \epsilon < r \)), \( \phi = g_r, \psi = (2\pi)^{-1} F \), in the limiting case \( \epsilon \downarrow 0 \), then reads
\[
(2) \quad T(r, f_w) = F^+_r(w) - F(w),
\]
where \( F^+_r \) is the least harmonic majorant of \( F \) in \( D(w, r) \); to be more precise,
\[
F^+_r(w) = (2\pi)^{-1} \int_{\partial D(w, r)} F(\xi) \frac{\partial}{\partial \nu} g_r(\xi) \, ds,
\]
where \(\frac{\partial}{\partial r}\) denotes the derivative in the inward-normal direction and \(ds\) is the element of arc length. Letting \(r \uparrow 1\) in (2) we obtain (1).

Suppose (II), and suppose then that
\[
\inf_{z \in D} F(z) = -\infty.
\]
Then there exists a sequence \(\{z_n\}\) in \(D\) such that
\[
|B_1(z_n)|^2 + |B_2(z_n)|^2 \to 0.
\]
By the same argument as in [2, p. 798], this contradicts our hypothesis (II). Therefore \(F\) is bounded from below and above in \(D\): \(-\infty < m \leq F \leq \log \sqrt{2}\). Thus,
\[
F - F^* \leq \log \sqrt{2} - m \quad \text{in } D,
\]
whence \(f \in \text{UBC}\) by (1).

REMARK. It follows from (II) that
\[
(IV) \quad \inf_{z \in D} \left( |B_1(z)|^2 + |B_2(z)|^2 \right) > 0.
\]
Our proof shows that (IV) \(\Rightarrow\) (III).

REFERENCES


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