WHICH AMALGAMS ARE CONVOLUTION ALGEBRAS?

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Abstract. We determine necessary and sufficient conditions on a locally compact abelian group $G$ for the amalgam $(L^p, l^q)(G)$ to be an algebra under convolution. If $q > 1$, $G$ must be compact; if $p < 1$, $G$ must be discrete. If $p \geq 1$ and $q \leq 1$, the amalgam is always an algebra.

1. Introduction. The amalgam of $L^p$ and $l^q$ is the space $(L^p, l^q)(G)$ consisting of all functions on a locally compact abelian group $G$ which are locally in $L^p$ and have $l^q$ behavior at infinity in the sense that the $L^p$-norms over certain compact subsets of $G$ form an $l^q$-sequence. (Precise definitions will be given in the next section.) We pose and answer the question: "Which of the amalgam spaces $(L^p, l^q)(G)$ are algebras under convolution?".

If $p = q$, then the amalgam $(L^p, l^q)(G)$ reduces to $L^p(G)$. For this case it is well known that $L^1(G)$ is always an algebra, and for $p > 1$ Żelazko [11] has proved that $L^p(G)$ is an algebra if and only if $G$ is compact. For the amalgams $(L^p, l^q)(G)$ with $p \geq 1$, $q \geq 1$, it is known that $(L^p, l^1)(G)$ is always an algebra, and we prove here that $(L^p, l^q)(G)$, $q > 1$, is an algebra if and only if $G$ is compact.

For indices smaller than 1, Żelazko [12] has shown that $L^p(G)$, $0 < p < 1$, is an algebra if and only if $G$ is discrete. Likewise, for amalgams, it turns out that if $(L^p, l^q)(G)$ is an algebra for $0 < p < 1$, then $G$ is discrete.

However, we also show that if $p \geq 1$ and $0 < q \leq 1$, then $(L^p, l^q)(G)$ is always an algebra. This provides a large class of new convolution algebras which, for $q < 1$, are $F$-algebras.

2. Amalgams. For functions of a real variable, Holland [4] defined the amalgam of $L^p$ and $l^q$ as the space $(L^p, l^q)$ of functions $f$ such that

$$\|f\|_{p,q} = \left[ \sum_{n=-\infty}^{\infty} \left( \int_{n}^{n+1} |f(x)|^p \, dx \right)^{q/p} \right]^{1/q} < \infty,$$

although certain special cases had been studied earlier starting with Wiener [10]. For functions on a locally compact abelian group $G$, amalgams have been defined and studied by Bertrandias, Datry and Dupuis [1], Stewart [8], and Busby and Smith [2]. In particular we follow the approach of [8] by using the structure theorem to write

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$G = R^a \times G_1$, where $a$ is a nonnegative integer and $G_1$ is a group which contains a compact open subgroup $H$. The Haar measure $m$ on $G_1$ is normalized so that $m(H) = 1$. Define $I = \{0,1\}^a \times H$ and $I_a = g_a + I$, where each $g_a$ is of the form $(n_1, \ldots, n_q, r)$ with $n_i \in \mathbb{Z}$ and the $t$'s being a transversal of $H$ in $G_1$, that is, $G_1 = \bigcup_j (t + H)$. We can then write $G$ as a disjoint union:

$$G = \bigcup_{a \in J} I_a.$$ 

In terms of this decomposition we define the amalgam $(L^p, l^q)(G)$ to be the space of functions $f$ which are locally in $L^p$ and are such that

$$\|f\|_{p,q} = \left[ \sum_{a \in J} \left( \int_{I_a} |f(x)|^p \, dx \right)^{q/p} \right]^{1/q} < \infty.$$ 

For $p = \infty$ we have

$$\|f\|_{\infty,q} = \left[ \sum_{a \in J} \sup_{x \in I_a} |f(x)|^q \right]^{1/q} < \infty \quad (q < \infty).$$

We shall often use the notation $f_a$ to mean the function which agrees with $f$ on $I_a$ and is 0 elsewhere. Then we can write

$$\|f\|_{p,q} = \left[ \sum_{a \in J} \|f_a\|^q_p \right]^{1/q}.$$ 

Notice that if $G$ is compact, then $(L^p, l^q)(G) = L^p(G)$. If $G$ is discrete, then we can take $I = \{0\}$ and so $(L^p, l^q)(G) = l^q(G)$.

Previously, amalgams have been studied for $p \geq 1$, $q \geq 1$, and in this case it is known that $\|f\|_{p,q}$ is a norm and $(L^p, l^q)(G)$ is a Banach space with dual $(L^{p'}, l^{q'})$, where $1/p + 1/p' = 1$, $1 \leq p, q < \infty$ [1]. However, if either $p$ or $q$ is less than 1, then $(L^p, l^q)$ is an $F$-space. To show this we establish that $\|f\|_{p,q}$ is a quasinorm by using the following inequalities [5, p. 158]:

$$\tag{2.1} (a + b)^p \leq 2^{p-1}(a^p + b^p) \quad (p > 1),$$

$$\tag{2.2} (a + b)^p \leq a^p + b^p \quad (0 < p < 1),$$

$$\tag{2.3} \|f + g\|_p \leq 2^{(1-p)/p}(\|f\|_p + \|g\|_p) \quad (0 < p < 1).$$

For $p \geq 1$, $q < 1$, we have

$$\|f + g\|_{p,q}^q = \sum_{a \in J} \|f + g\|_{a}^q \leq \sum_{a \in J} (\|f_a\|_p + \|g_a\|_p)^q \leq \sum_{a \in J} \|f_a\|_p^q + \sum_{a \in J} \|g_a\|_p^q \quad \text{[by (2.2)]}$$

$$= \|f\|_{p,q}^q + \|g\|_{p,q}^q.$$ 

Then (2.1) gives

$$\tag{2.4} \|f + g\|_{p,q} \leq 2^{(1-q)/q}(\|f\|_{p,q} + \|g\|_{p,q}) \quad (p \geq 1, q < 1).$$
Similarly, using (2.1), (2.2) and (2.3) we obtain

\[(2.5) \quad \|f + g\|_{p,q} \leq 2^{1-p/p} \|f\|_{p,q} + \|g\|_{p,q} \quad (p < 1, q \geq 1),\]

\[(2.6) \quad \|f + g\|_{p,q} \leq 2^{1-p/p} 2^{1-q/q} \|f\|_{p,q} + \|g\|_{p,q} \quad (p < 1, q < 1).\]

The inequalities (2.4), (2.5) and (2.6) show that when either \(p\) or \(q\) is less than 1, \((L^p, l^q)(G)\) is a quasi-normed space which is therefore locally bounded [5, p. 159] and hence an \(F\)-space [7].

The following relations were given in [8] for \(p, q \geq 1\) but continue to hold when either index is less than 1:

\[(2.7) \quad (L^p, l^{q_1}) \subset (L^p, l^{q_2}) \quad (0 < p \leq \infty, 0 < q_1 \leq q_2 \leq \infty),\]

\[(2.8) \quad (L^{p_1}, l^{q}) \subset (L^{p_1}, l^{q_2}) \quad (0 < p_1 \leq p \leq \infty, 0 < q \leq \infty),\]

\[(2.9) \quad (L^p, l^p) = L^p \quad (0 < p \leq \infty),\]

\[(2.10) \quad (L^p, l^q) \subset L^p \cap L^q \quad (0 < q \leq p \leq \infty),\]

\[(2.11) \quad L^p \cup L^q \subset (L^p, l^q) \quad (0 < p \leq q \leq \infty).\]

Associated with (2.7) and (2.8) are the inequalities

\[(2.12) \quad \|f\|_{p,q_1} \leq \|f\|_{p,q_2} \quad (0 < p \leq \infty, 0 < q_1 \leq q_2 \leq \infty),\]

\[(2.13) \quad \|f\|_{p_1,q} \leq \|f\|_{p,q} \quad (0 < p_1 \leq p \leq \infty, 0 < q \leq \infty).\]

3. The case \(p \geq 1, q \geq 1\). If we combine Young’s Inequality for amalgams [1 or 2] with the inequality (2.13), we get

\[\|f \ast g\|_{p,1} \leq C\|f\|_{1,1}\|g\|_{p,1} \leq C\|f\|_{p,1}\|g\|_{p,1},\]

and this shows that \((L^p, l^1)(G)\) is a Banach algebra for \(p \geq 1\) and for any \(G\). (In fact, it is a Segal algebra [6, 1]. The classic special case is the Wiener algebra \((L^\infty, l^1) \cap C(R)\) as studied by Goldberg [3].)

We now consider the case \(p \geq 1, q > 1\). Here \((L^p, l^q)(G)\) is a Banach space, so if it is a topological algebra, then it is well known that there is an equivalent submultiplicative norm \(\|\cdot\|\) such that \(\|f\| \geq \|f\|_{p,q}\). The proof of the following theorem is adapted from Urbanik [9].

**Theorem 1.** The amalgam \((L^p, l^q)(G), p \geq 1, q > 1\), is a topological algebra if and only if \(G\) is compact.

**Proof.** If \(G\) is compact, then \((L^p, l^q)(G) = L^p(G)\) which is known to be an algebra [11].

Now suppose that \(A = (L^p, l^q)(G)\) is a topological algebra. We first show that \(A \neq \text{rad}(A)\). Let \(V\) be a symmetric, compact neighborhood of 0 in \(G\) and let \(\chi\) be the characteristic function of \(V + V\). Then, for \(x \in V\), we have

\[\chi^{n+1}(x) = \int_G \chi(x - y) \chi^n(y) \, dy \geq \int_V \chi^n(y) \, dy,\]
where \( \chi^n \) means the \( n \)-fold convolution of \( \chi \). By induction it follows that \( \chi^{n+1}(x) \geq [m(V)]^n \) when \( x \in V \). Putting \( c = 1/\|x\| \), we obtain
\[
\|\chi^n\| \geq c\|\chi^{n+1}\| \geq c\|\chi^{n+1}\|_{p,q}
\]
\[
= c \left[ \sum_{\alpha \in J} \left[ \int_{I_{\alpha}} |\chi^{n+1}|^p \right]^{q/p} \right]^{1/q} \geq c \left[ \sum_{\alpha \in J} \left[ \int_{I_{\alpha} \cap V} |m(V)|^p \right]^{q/p} \right]^{1/q}
\]
\[
\geq c \left[ \sum_{\alpha \in J} \left[ \int_{I_{\alpha} \cap V} m(V) \right]^{nq} \right]^{q/p} = c \left[ \sum_{\alpha \in J} m(V)^{nq} m(V \cap I_{\alpha})^{q/p} \right]^{1/q}
\]
\[
\geq c [m(V)]^n \left[ \sum_{\alpha \in J} [m(V \cap I_{\alpha})]^{q/p} \right]^{1/q} > 0.
\]
Thus
\[
\lim_{n \to \infty} \|\chi^n\|^{1/n} \geq m(V) > 0,
\]
and so \( \chi \notin \text{Rad}(A) \). Therefore there is a nontrivial multiplicative linear functional \( T \) on \( A \). Since \( T \) is continuous there exists a function \( f \) in \( (L^{p'}, l^{q'}) \) with \( \|f\|_{p',q'} > 0 \) such that
\[
T(\phi) = \int_G f(x) \phi(x) \, dx \quad (\phi \in (L^p, l^q)).
\]
Since \( T \) is multiplicative, for every \( \phi, \psi \in (L^p, l^q) \) we have
\[
\int_G \int_G f(x+y) \phi(x) \psi(y) \, dx \, dy = \int_G f(x) \left[ \int_G \phi(x-y) \psi(y) \, dy \right] \, dx
\]
\[
= \int_G f(x) (\phi \star \psi)(x) \, dx = T(\phi \star \psi) = T(\phi) T(\psi)
\]
\[
= \int_G \int_G f(x) f(y) \phi(x) \psi(y) \, dx \, dy.
\]
This shows that
\[
f(x+y) = f(x) f(y) \quad (l. \text{ a.e.}).
\]
Now we use the fact that \( (L^p, l^q) \) has an equivalent translation-invariant norm \( \| \cdot \|_{p,q}^\# \) [1, Proposition VIII]:
\[
c'' \|f\|_{p,q}^\# \leq \|f\|_{p,q} \leq c' \|f\|_{p,q}^\#.
\]
It follows that translation is a bounded operator on \( (L^p, l^q) \):
\[
\|f_{x}\|_{p,q} \leq K \|f\|_{p,q},
\]
where \( f_{x}(x) = f(x+y) \). Applying this to \( (L^{p'}, l^{q'}) \), we have
\[
\|f\|_{p',q'} \leq K \|f_{x}\|_{p',q'} = K \left[ \sum_{\alpha \in J} \left[ \int_{I_{\alpha}} |f(x+y)|^{p'} \, dx \right]^{q'/p'} \right]^{1/q'}
\]
\[
= K \|f(y)\| \left[ \sum_{\alpha \in J} \left[ \int_{I_{\alpha}} |f(x)|^{p'} \, dx \right]^{q'/p'} \right]^{1/q'} = K \|f(y)\| \|f\|_{p',q'}.
\]
Since \( \|f\|_{p',q'} \neq 0 \), we conclude that \( |f(y)| \geq 1/K \) on \( G \). This shows that the constant functions belong to \( (L^p, l^q) \). Since \( q > 1 \), we have \( q' \neq \infty \), and so we can write

\[
\infty > \|1\|_{p',q'} = \sum_{a \in J} \left[ m(I_a) \right]^{q'/p'} = \sum_{a \in J} m(I_a) = m(G).
\]

Thus \( G \) is compact.

4. The case \( p < 1 \). For any nondiscrete group \( G \), Želazko [12] has constructed two functions \( f \) and \( g \) in \( L^p(G) \) whose convolution \( f \ast g \) is infinite on a set of positive measure. These functions \( f \) and \( g \) have compact support and so they belong to \( (L^p, l^q)(G) \) for any \( q > 0 \). Therefore we have the following theorem.

**Theorem 2.** If the amalgam \( (L^p, l^q)(G) \), \( 0 < p < 1 \), is a topological algebra, then \( G \) is discrete.

In order to give a converse, we first note that if \( G \) is discrete, then \( (L^p, l^q)(G) = l^q(G) \). If \( q \leq 1 \), then \( l^q(G) \) is a topological algebra [12]. If \( q > 1 \) and \( l^q(G) \) is an algebra, then \( G \) is both discrete and compact, hence finite. And if \( G \) is finite, \( l^q(G) \) is always an algebra. We have thus proved the following corollaries of Theorem 2.

**Corollary 1.** The amalgam \( (L^p, l^q)(G) \), \( 0 < p < 1 \), \( 0 < q \leq 1 \), is a topological algebra if and only if \( G \) is discrete.

**Corollary 2.** The amalgam \( (L^p, l^q)(G) \), \( 0 < p < 1 \), \( q > 1 \), is a topological algebra if and only if \( G \) is finite.

5. The case \( p \geq 1, q \leq 1 \). We have seen that for \( (L^p, l^q)(G) \) to be a topological algebra, \( G \) must be compact if \( q > 1 \) and \( G \) must be discrete if \( p < 1 \). By contrast with this situation, we show that in the remaining case \( (p \geq 1, q \leq 1) \) \( (L^p, l^q)(G) \) is always an algebra.

**Theorem 3.** The amalgam \( (L^p, l^q)(G) \), \( p \geq 1, 0 < q \leq 1 \), is a topological algebra under convolution for any locally compact abelian group \( G \). Moreover, for \( f \) and \( g \) in \( (L^p, l^q)(G) \) we have

\[
\|f \ast g\|_{p,q} \leq 2^{a/q} \|f\|_{p,q} \|g\|_{p,q},
\]

where \( a \) is given by the structure theorem as in §2.

**Proof.** We first observe that, in the notation introduced in §2,

\[
(f \ast g)(x) = \sum_{a \in J} \sum_{\beta \in J} (f_a \ast g_\beta)(x).
\]

Therefore, for any \( \gamma \in J \), Minkowski’s Inequality gives

\[
\|(f \ast g)(\gamma)\|_p = \| \sum_{a} \sum_{\beta} (f_a \ast g_\beta)(\gamma) \|_p \leq \sum_{a} \sum_{\beta} \|(f_a \ast g_\beta)(\gamma)\|_p \leq \sum_{a} \sum_{\beta} \|(f_a \ast g_\beta)\|_p.
\]
where we use the notation $\gamma \subset \alpha + \beta$ to mean that we sum over all indices $\alpha$ and $\beta$ in $J$ such that $I_\gamma \subset I_\alpha + I_\beta$. Now we apply Young’s Inequality for $L^p$-spaces to write
\[
\|(f \ast g)_\gamma\|_p \leq \sum_{\alpha \beta} \|f_\alpha\|_1 \|g_\beta\|_p, \quad \gamma \subset \alpha + \beta
\]
Using the fact that $q \leq 1$ together with (2.2), we have
\[
\|(f \ast g)_\gamma\|_p^q \leq \left( \sum_{\alpha \beta} \|f_\alpha\|_1 \|g_\beta\|_p \right)^q \leq \sum_{\alpha \beta} \|f_\alpha\|_1 \|g_\beta\|_p^q, \quad \gamma \subset \alpha + \beta
\]
and so
\[
\|f \ast g\|_{p,q}^q = \sum_{\gamma} \|(f \ast g)_\gamma\|_p^q \leq \sum_{\gamma} \sum_{\alpha} \sum_{\beta} \|f_\alpha\|_1 \|g_\beta\|_p^q \leq 2^\alpha \|f\|_{1,q} \|g\|_{p,q}^q.
\]
Note the presence of the factor $2^\alpha$. This is because, for fixed $\alpha$ and $\gamma$, there are $2^\alpha$ $I_\beta$’s such that $I_\gamma \subset I_\alpha + I_\beta$, and as we sum over $\gamma$ each such $\beta$ occurs exactly $2^\alpha$ times. Finally, taking $q$th roots, we obtain
\[
\|f \ast g\|_{p,q} \leq 2^{a/q}\|f\|_{1,q}\|g\|_{p,q} \quad \text{[by (2.13)].}
\]
This completes the proof.

In view of the fact, established in §2, that $(L^p, l^q)(G)$ is an $F$-space whenever either index is less than 1, Theorem 3 says that $(L^p, l^q)(G)$, $p \geq 1$, $q \leq 1$, is an $F$-algebra.

Notice from the proof of Theorem 3 that
\[
\|f \ast g\|_{p,q} \leq 2^{a/q}\|f\|_{1,q}\|g\|_{p,q},
\]
and this shows that $(L^p, l^q)(G)$ is an ideal in the algebra $(L^1, l^q)(G)$.

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