ON FIXED POINTS OF LINEAR CONTRACTIONS

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Abstract. It is shown that a weakly closed convex semigroup of linear contractions on a separable Hilbert space has a common fixed point other than 0 if the operator 0 is not in the semigroup.

We prove a theorem on existence of common fixed points for certain convex semigroups of linear operators on Banach spaces. The special case where the semigroup is a group follows easily from Kakutani's well-known theorem [4, 5, 6] and also, as discussions with P. Milman revealed, from the work of Brodskii and Milman [1]. Similarly, in the case where the semigroup is commutative, our result is a corollary of a special case of the Markov-Kakutani theorem [3, 4]. Nonetheless, it appears that the results and corollaries given below have not been noticed before. Corollary 4, for example, gives a sufficient condition that $\bigcup_{n=N}^{\infty} \{ A^n \}$ be the same for all $N$.

The applications of the fixed-point theorem that we consider concern operators on Hilbert space, but it seems worthwhile to state the theorem more generally.

Theorem 1. Let $X$ be a strictly convex reflexive Banach space, and let $\mathcal{S}$ be a weak operator closed separable convex semigroup of linear contractions on $X$. Then the operators in $\mathcal{S}$ have a common fixed point other than 0 if and only if the operator 0 is not in $\mathcal{S}$.

Proof. Clearly, if the operator 0 is in $\mathcal{S}$, then the only common fixed point is 0.

To prove the converse first recall that $(T_n) \to T$ in the weak operator topology if and only if $\phi(T_n x) \to \phi(T x)$, for each $\phi \in X^*$ and $x \in X$. We require the fact that the unit ball of $B(X)$ is weak operator compact; this can be proven as in the better-known case of Hilbert space. (That is, consider the Cartesian product of the closed balls of radius $\|x\|$ in $X$, indexed by $X$, where each ball is given the weak topology).

Let $\{ T_n \}_{n=1}^{\infty}$ be a countable weak operator dense subset of $\mathcal{S}$; it obviously suffices to find a common fixed point for the $\{ T_n \}$. Let

$$T = \sum_{n=1}^{\infty} \frac{1}{2^n} T_n;$$

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this series converges in the norm topology (hence also in the weak operator topology) of \( \mathcal{B}(X) \), and the closed convexity of \( \mathcal{S} \) implies \( T \in \mathcal{S} \). Now \( T \) defines a mapping of \( \mathcal{S} \) into itself by \( T(S) = TS \) for \( S \in \mathcal{S} \) (\( \mathcal{S} \) is a semigroup). Since \( \mathcal{S} \) is a compact convex set, Schauder's fixed point theorem yields an operator \( S_0 \in \mathcal{S} \) such that \( TS_0 = S_0 \). Choose \( x \in \mathcal{S} \) such that \( S_0 x \neq 0 \). Then
\[
\sum_{n=1}^{\infty} \frac{1}{2^n} T_n S_0 x = S_0 x.
\]
For each \( n_0 \),
\[
\left\| \sum_{n \neq n_0} \frac{1}{2^n} T_n S_0 x + \frac{1}{2^{n_0}} T_{n_0} S_0 x \right\| = \| S_0 x \|,
\]
\[
\left\| \sum_{n \neq n_0} \frac{1}{2^n} T_n S_0 x \right\| \leq \left( \sum_{n \neq n_0} \frac{1}{2^n} \right) \| S_0 x \|,
\]
and
\[
\left\| \frac{1}{2^{n_0}} T_{n_0} S_0 x \right\| \leq \frac{1}{2^{n_0}} \| S_0 x \|
\]
implies that the above inequalities are equations, so the strict convexity of \( \mathcal{S} \) implies that \( T_{n_0} S_0 x \) is a multiple of \( \sum_{n+n_0} T_n S_0 x / 2^n \). Hence, \( T_{n_0} S_0 x \) is a multiple of \( S_0 x \).
(Recall that \( \mathcal{S} \) strictly convex means that \( \| x_1 + x_2 \| = \| x_1 \| + \| x_2 \| \) implies \( \{ x_1, x_2 \} \) is linearly dependent). Thus, \( T_n S_0 x \) is a multiple of \( S_0 x \) for every \( n \). But \( \{ \lambda_n \} \), complex numbers, satisfying \( \sum_{n=1}^{\infty} \lambda_n / 2^n = 1 \) and \( |\lambda_n| \leq 1 \) for all \( n \) implies \( \lambda_n = 1 \) for all \( n \), so \( T_n S_0 x = S_0 x \) for all \( n \). Therefore, \( S_0 x \) is a common fixed point for \( \{ T_n \} \) and, hence, for \( \mathcal{S} \).

**Remark.** As the referee has kindly pointed out, the above proof is similar to a proof given by R. E. Bruck, Jr., *Properties of fixed-point sets of nonexpansive mappings in Banach spaces*, Trans. Amer. Math. Soc. 179 (1973), 251–262.

**Corollary 1.** A weakly closed convex semigroup of contractions on a separable Hilbert space has a common fixed point other than 0 if and only if it does not contain the operator 0.

**Proof.** A Hilbert space satisfies all the hypotheses on \( \mathcal{S} \) in Theorem 1. Also, the unit ball of operators on a separable Hilbert space is a separable metrizable space in the weak operator topology, so every semigroup of contractions is separable.

For the next two corollaries let \( \mathcal{S} \) be a weakly closed convex semigroup of contractions on a separable Hilbert space.

**Corollary 2.** Let \( M \) denote the set of common fixed points of members of \( \mathcal{S} \); then \( \mathcal{S} \) contains the orthogonal projection onto \( M \).

**Proof.** As is well known, \( \| T \| \leq 1 \) and \( T x = x \) implies \( T^* x = x \) (begin an orthonormal basis with \( x/\| x \| \) and represent \( T \) with respect to it). Thus, \( M \) reduces every operator in \( \mathcal{S} \). Now \( \mathcal{S} \cap M^\perp \) is a weakly closed convex semigroup of contractions on \( M^\perp \). Since the only common fixed point of \( \mathcal{S} \cap M^\perp \) is \( \{ 0 \} \), Corollary 1
implies that the 0 operator is in \( \mathcal{S} \cap \mathcal{M}^\perp \). Let \( P \in \mathcal{S} \) be such that \( P|\mathcal{M}^\perp = 0 \); since \( P|\mathcal{M} \) is the identity, \( P \) is the projection on \( \mathcal{M} \).

**Corollary 3.** If \( \mathcal{S} \) is not the semigroup consisting only of the identity, then some operator in \( \mathcal{S} \) has nontrivial nullspace.

**Proof.** By Corollary 2, if no operator in \( \mathcal{S} \) has nullspace, then the set of common fixed points is the entire space. The next result is a corollary of Theorem 1 in some cases but not in all. The proof, however, is contained in that of Theorem 1.

**Theorem 2.** If \( \mathcal{S} \) is a weak operator closed bounded convex set of linear operators on a reflexive space and \( 0 \in \mathcal{S} \), then 1 is an eigenvalue of every operator \( T \) with the property that \( S \in \mathcal{S} \) implies \( TS \in \mathcal{S} \).

**Proof.** Let \( T \) be as stated. By Schauder's theorem, \( TS_0 = S_0 \) for some \( S_0 \in \mathcal{S} \). Choose \( x \) such that \( S_0x \neq 0 \); then \( TS_0x = S_0x \), so 1 is an eigenvalue of \( T \).

**Corollary 4.** If \( A \) is an injective operator on Hilbert space, and if there is a \( k \) such that \( \|(1 + A)^n\| \leq k \) for every positive integer \( n \), then the weakly closed linear span of \( \{A^n: n \geq N\} \) is the same for all nonnegative integers \( N \).

**Proof.** Let \( T = 1 + A \) and let \( \mathcal{S} \) be the weakly closed convex hull of \( \{T^n: n \geq 1\} \). Since \( A \) has no nullspace, \( T \) has no fixed points other than 0. By Theorem 2, \( 0 \in \mathcal{S} \). Thus, given any weak operator neighborhood \( \mathcal{W} \) of 0 there is a collection of nonnegative numbers \( \{ \lambda_j \}_{j=1}^m \) such that \( \sum_{j=1}^m \lambda_j = 1 \) and \( \sum_{j=1}^m \lambda_j T^j \in \mathcal{W} \). Then \( \sum_{j=1}^m \lambda_j T^j \) has the form \( 1 + \sum_{j=1}^m \lambda_j p_j(A) \) for suitable polynomials \( p_j \) without constant terms. It follows that 1 is in the weak closure of the linear span of \( \{A^n: n \geq 1\} \). Thus, \( A \) is also in the weak closure of the linear span of \( \{A^n: n \geq 2\} \) (multiplication is separately weakly continuous in each variable), and the corollary follows by a trivial induction.

**References**


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