A HOLOMORPHIC FUNCTION WITH WILD BOUNDARY BEHAVIOR

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To the memory of Darja

Abstract. Let $B$ be the open unit ball in $\mathbb{C}^N$, $N > 1$. It is known that if $f$ is a function holomorphic in $B$, then there are $x \in \partial B$ and an arc $\Lambda$ in $B \cup \{x\}$, with $x$ as one endpoint along which $f$ is constant. We prove

**Theorem.** There exist an $r > 0$ and a function $f$ holomorphic in $B$ with the property that, if $x \in \partial B$ and $\Lambda$ is a path with $x$ as one endpoint, such that $\Lambda \setminus \{x\}$ is contained in the open ball of radius $r$ which is contained in $B$ and tangent to $\partial B$ at $x$, then $\lim_{z \in \Lambda, z \to x} f(z)$ does not exist.

We denote by $B$ the open unit ball in $\mathbb{C}^N$, $N > 1$. For each $x \in \partial B$ and $r$, $0 < r < 1$, let $D(x, r)$ be the open ball of radius $r$, contained in $B$ and tangent to $\partial B$ at $x$. We prove the following

**Theorem.** There exist an $r > 0$ and a function $f$ holomorphic in $B$ such that if $x \in \partial B$ and $\Lambda$ is a path contained in $D(x, r)$, except for its endpoint $x$, then $\lim_{z \in \Lambda, z \to x} f(z)$ does not exist.

It is known that $r$ in the Theorem has to be strictly smaller than $1$ [2]; whether or not it can be arbitrarily close to $1$ is an open question.

For each $x \in \partial B$ and each $\rho$, $0 < \rho < 1$, let $H(\rho x)$ be the real hyperplane through $\rho x$, tangent to $\partial B$ at $\rho x$. If $R > 0$ let

$$W(\rho x, R) = \{y \in H(\rho x) : |y - \rho x| < R\}.$$ 

Thus $W(\rho x, R)$ is the relatively open ball in $H(\rho x)$ of radius $R$ centered at $\rho x$.

**Lemma 1.** There is an $r > 0$ with the following property: let $0 < a < 1$; there exists $L \in \mathbb{N}$, numbers $\rho_i$, $1 \leq i \leq L$, $a < \rho_1 < \cdots < \rho_L < \rho_{L+1} = 1$, and numbers $R_i > 0$, $1 \leq i \leq L$, such that $W(x, R_i) \subset \rho_{i+1}B$ for every $x \in \partial(\rho_iB)$, $1 \leq i \leq L$, and for each $l$, $1 \leq l \leq L$, there is a finite set $T_l \in \partial(\rho_iB)$ such that

(i) $W(x, R_i) \cap H(y) = \emptyset$ whenever $x, y \in T_l, x \neq y, 1 \leq l \leq L$;

(ii) given any $y \in \partial B$ there exist $l$, $1 \leq l \leq L$, and $z \in T_l$ such that if $\Lambda$ is a path joining a point in $aB$ with $y$, such that $\Lambda \setminus \{y\} \subset D(y, r)$, then $\Lambda$ meets $W(z, R_l)$.

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Lemma 2. Let $0 < a < 1$. Let $L$, $p_i$, $R_i$, and $T_i$, $1 \leq l \leq L$, be as in Lemma 1. Given $\epsilon > 0$ and $C < \infty$ there is a polynomial $P$ such that

(i) $\text{Re} P > C$ on $\bigcup_{l=1}^{L-1} \bigcup_{x \in T_l} W(x, R_l)$;

(ii) $|P| < \epsilon$ on $aB$.

Proof. Choose $p_0^i: a < p_1^i < p_2^i < \cdots < p_L^i < p_{L+1}^i < 1$ such that for each $l$, $1 \leq l \leq L$, $W(x, R_l) \subset c p_l^i B (x \in \partial(p_l^i B))$. Fix $l$, $1 \leq l \leq L$, and denote $W_l = \bigcup_{x \in T_l} W(x, R_l)$. If $\delta_l > 0$ and $C_l < \infty$, then, by Lemma 1(i), one can prove, similarly to the proof of Theorem 4 in [1], that there is a polynomial $P_l$ such that $|P_l| < \delta_l$ on $p_l^i B$ and $\text{Re} P_l > C_l$ on $W_l$. If we choose $\delta_l$ and $C_l$ properly, then $P = \sum_{l=1}^{L} P_l$ will have all the required properties. This completes the proof.

Proof of the Theorem. By Lemmas 1 and 2 there exist an $r > 0$, a sequence $a_n$, $0 < a_1 < \cdots < 1$, lim $a_n = 1$, and a sequence of sets $W_n$, $W_n \subset a_{n+1} B - a_n B$, such that if $n \in \mathbb{N}$, $x \in \partial B$, and $\Lambda$ is a path joining a point in $a_n B$ with $x$, which satisfies $\Lambda - \{x\} \subset D(x, r)$, then $\Lambda$ meets $W_n$; moreover, for each $n \in \mathbb{N}$, $\delta_n > 0$, and $C_n < \infty$ there is a polynomial $P_n$ such that $|P_n| < \delta_n$ on $a_n B$ and $\text{Re} P_n > C_n$ on $W_n$. If the sequence $C_n$ is chosen inductively to increase to $+\infty$ fast enough, and if the sequence $\delta_n$ is chosen to decrease to 0 fast enough, then the series $\sum_{n=1}^{\infty} P_n$ converges uniformly on compacta in $B$ to a function $f$ holomorphic in $B$ with the property: if $x \in \partial B$ and $\Lambda$ is a path with $x$ as one endpoint which satisfies $\Lambda - \{x\} \subset D(x, r)$, then

$$\limsup_{z \in \Lambda : z \to x} \text{Re} f(z) = +\infty, \quad \liminf_{z \in \Lambda : z \to x} \text{Re} f(z) = -\infty.$$ 

This completes the proof.

To prove Lemma 1, we first prove three lemmas.

Lemma 3. Let $x, y \in \partial B$ and $|x - y| > 2R/\rho$, where $0 < \rho < 1$ and $R > 0$. Then $W(\rho x, R) \cap H(\rho y) = \varnothing$.

Proof. Suppose $z \in W(\rho x, R) \cap H(\rho y)$. Then $|z|^2 < \rho^2 + R^2$; i.e., $z \in W(\rho y, R)$ and, consequently, $\rho |x - y| < |z - \rho x| + |z - \rho y| < 2R$, a contradiction.

Lemma 4. Let $0 < r < 1$, $0 < \rho < 1$, and $0 < P < 2^{1/2}$. Suppose $x, y \in \partial B$ and $|x - y| < P(1 - \rho)^{1/2}$. Then $x$ and $y$ both lie on the same side of $H(\rho x)$. Moreover, $H(\rho x) \cap D(y, r) \subset W(\rho x, Q(1 - \rho)^{1/2})$, where $Q = (1 - r)P + (2r)^{1/2}$.

Proof. The first statement follows from the fact that $P < 2^{1/2}$, which implies that $|x - y| < (2(1 - \rho))^{1/2}$. Suppose $H(\rho x) \cap D(y, r)$ is not empty. Write $y = \alpha x + w$, $\rho < \alpha \leq 1$, $|w|^2 + \alpha^2 = 1$. The center of $D(y, r)$ is at a distance of $|\rho - (1 - r)\alpha|$ from $H(\rho x)$ and at a distance of $(1 - r)(1 - \alpha^2)^{1/2}$ from $\mathbb{R}x$. Consequently, $H(\rho x) \cap D(y, r) \subset W(\rho x, R)$, where

$$R = (1 - r)(1 - \alpha^2)^{1/2} + \left[ r^2 - [\rho - (1 - r)\alpha]^2 \right]^{1/2}$$

$$= (1 - r)(1 - \alpha^2)^{1/2} + \left[ (1 - \rho - (1 - r)(1 - \alpha)) \cdot [2r - (1 - \rho) + (1 - r)(1 - \alpha)] \right]^{1/2}.$$
Since $|x - y| < P(1 - \rho)^{1/2}$, we have $(1 - \alpha)^2 + (1 - \alpha^2) < P^2(1 - \rho)$; hence $1 - \alpha < (1 - \rho)P^{2/2}$, and, consequently,

$$R \leq (1 - r)(1 - \rho)^{1/2}(P/2^{1/2}) \cdot 2^{1/2} + (2r(1 - \rho))^{1/2}.$$  

This completes the proof.

**Lemma 5.** Let $p \in \mathbb{N}$ and $x \in \partial B$. There exist a neighbourhood $U \subset \partial B$ of $x$, an $r_0 > 0$, and $M \in \mathbb{N}$ such that, for any $r, 0 < r < r_0$, there are finite sets $S_m \subset U$, $1 \leq m \leq M$, such that $U \subset \bigcup_{m=1}^{M} \bigcup_{y \in S_m}(y + rB)$ and $|y - z| \geq pr$ whenever $y, z \in S_m, y \neq z, 1 \leq m \leq M$.

**Proof.** Part 1. We prove the following. Let $W \subset \mathbb{R}^{2N - 1}$ be a bounded set and let $k \in \mathbb{N}$. There is a $\mu = \mu(k, N) \in \mathbb{N}$ such that, given any $r > 0$, there are finite sets $T_m \subset \mathbb{R}^{2N - 1}, 1 \leq m \leq \mu$, such that $W \subset \bigcup_{m=1}^{\mu} \bigcup_{y \in T_m}(y + rB)$ (in this part $B$ is the open unit ball in $\mathbb{R}^{2N - 1}$) and $|y - z| \geq kr$ whenever $y, z \in T_m, y \neq z, 1 \leq m \leq \mu$.

To do this put $L = 2N - 1$, choose $q \in \mathbb{N}$ such that $q > kL^{1/2}$, and put $\mu = qL$. Let $r > 0$. Define $S \subset \mathbb{R}^L$ by

$$S = \{kr(s_1, s_2, \ldots, s_L): s_i \in \mathbb{Z}, 1 \leq i \leq L\}.$$  

Observe that $|y - z| \geq kr$ whenever $y, z \in S, y \neq z$. Further, let $P$ be the set of $\mu$ points in the cube $\{t \in \mathbb{R}^L: 0 \leq t_i \leq kr, 1 \leq i \leq L\}$, defined by

$$P = \{(kr/q)(s_1, s_2, \ldots, s_L): s_i \in \mathbb{Z}, 1 \leq s_i \leq q, 1 \leq i \leq L\}.$$  

There are $\mu$ sets of the form $y + S$, where $y \in P$, and they have the following properties:

(a) if $y \in P$ and $z, w \in y + S, z \neq w$, then $|w - z| \geq kr$;

(b) $\mathbb{R}^L = \bigcup_{y \in P} \bigcup_{z \in y + S}(z + K)$,

where $K$ is the cube $\{t \in \mathbb{R}^L: |t_i| < kr/q, 1 \leq i \leq L\}$.

Since $q > kL^{1/2}$, it follows that $kr/q < rL^{-1/2}$; hence $K \subset rB$, which implies $\mathbb{R}^L = \bigcup_{y \in P} \bigcup_{z \in y + S}(z + rB)$. Now the assertion follows from the boundedness of $W$.

**Part 2.** There exist an open neighbourhood $U' \subset \partial B$ of $x$, an open neighbourhood $V \subset \mathbb{R}^{2N - 1}$ of $0$, a constant $c > 0$, and a map $\Psi$ from $V$ onto $U'$ such that

$$(1/c)|u - v| < |\Psi(u) - \Psi(v)| < c|u - v| \quad (u, v \in V).$$

Let $U \subset U'$ be a compact neighbourhood of $x$. The statement of the lemma now follows easily from Part 1. This completes the proof.

**Proof of Lemma 1.** It is enough to prove the following. Let $x \in \partial B$. There exist $M \in \mathbb{N}, r > 0$, a neighborhood $U \subset \partial B$ of $x$, and $a, 0 < a < 1$, such that the following holds: Given any $\rho_1, a < \rho_1 < 1$, there exist $R > 0$ and $\rho_m, 1 < m \leq M + 1, \rho_1 < \rho_2 < \cdots < \rho_M < \rho_{M+1} < 1$, such that $\bar{W}(y, r) \subset \rho_{m+1}B$ ($y \in \partial(\rho_mB), 1 \leq m \leq M$) and, for each $m, 1 \leq m \leq M$, there is a finite set $S_m \subset \partial(\rho_mB)$ such that

(i) $\bar{W}(y, R) \cap H(z) = \emptyset$ whenever $y, z \in S_m, y \neq z, 1 \leq m \leq M$;

(ii) given any $y \in U$ there exist $m, 1 \leq m \leq M$, and $z \in S_m$ such that if $\Lambda$ is a path joining a point in $\rho_1B$ with $y$, where $\Lambda - \{y\} \subset D(y, r)$, then $\Lambda$ meets $W(z, R)$.
To prove this let $p = 9$ and let $U$, $r_0$, and $M$ be as in Lemma 5. Choose $P$, $0 < P < 2^{1/2}$, and $r > 0$ such that

(1) \[ Q = (1 - r) P + (2r)^{1/2} < (2M)^{-1/2} \]

and

(2) \[ 8(1 - r) + (2r)^{1/2}/P < p. \]

Note that, by (2),

(3) \[ 8Q/P < p. \]

Choose $a < 1$ so close to 1 that

(4) \[ 1/2 < a, \]

(5) \[ 1 - a < r, \]

and

(6) \[ P((1 - a)/2)^{1/2} < r_0. \]

Let $a < \rho_1 < 1$. Set $\vartheta = (1 - \rho_1)/(2M)$ and let

$\rho_m = \rho_1 + (m - 1) \vartheta \quad (1 \leq m \leq M + 1).$

Put $R = Q(1 - \rho_1)^{1/2}$. By (1) and (4),

$R < (2M)^{-1/2}(1 - \rho_1)^{1/2} = \vartheta^{1/2} = (\rho_{m+1} - \rho_m)^{1/2}$

so $W(y, R) \subseteq \rho_{m+1} B (x \in \partial(\rho_mB), 1 \leq m \leq M)$.

Now let $\varepsilon = P(1 - \rho_1)^{1/2} \cdot 2^{-1/2}$. By (6), $\varepsilon < r_0$. Furthermore, since $(1 - \rho_1)/2 < 1 - \rho_m (1 \leq m \leq M)$, it follows that $\varepsilon < P(1 - \rho_m)^{1/2} (1 \leq m \leq M)$. By (5), $1 - \rho_m < r (1 \leq m \leq M)$; since $P < 2^{1/2}$, it follows, by Lemma 4, that if $y, z \in \partial B$, $|y - z| < \varepsilon$, then both $y$ and $z$ lie on the same side of $H(\rho_m y), 1 \leq m \leq M$, and furthermore,

if $1 \leq m \leq M$ and if $y, z \in \partial B, |y - z| < \varepsilon$, then every path

(7) \[ \Lambda, \text{which joins a point in } \rho_mB \text{with } z \text{ and satisfies } \Lambda - \{z\} \subseteq D(z, r), \text{meets } W(\rho_m y, Q(1 - \rho_m)^{1/2}) \subseteq W(\rho_m y, R). \]

Furthermore, since $\varepsilon < r_0$, it follows by Lemma 5 that there are finite sets $T_m \subseteq \partial B, 1 \leq m \leq M$, such that

(a) $|y - z| \geq p\varepsilon$ whenever $y, z \in T_m, y \neq z, 1 \leq m \leq M$;

(b) $U \subseteq \bigcup_{m=1}^M \bigcup_{y \in T_m} (y + \varepsilon B)$.

Define $S_m = \rho_m T_m (1 \leq m \leq M)$.

Suppose $1 \leq m \leq M$ and $y, z \in T_m, y \neq z$. By (a) and (3) it follows that

$|y - z| \geq p\varepsilon > 8(Q/P)\varepsilon = 8(Q/P) \cdot P((1 - \rho_1)/2)^{1/2}$

$= 8Q \cdot 2^{-1/2}(1 - \rho_1)^{1/2} > 4R.$

By (4), $\rho_m > 1/2$ so, by Lemma 3, $W(\rho_m y, R) \cap H(\rho_m z) = \emptyset$. This proves (i).

Suppose $y \in U$. By (b) there exist $m, 1 \leq m \leq M$, and $z \in T_m$ such that $|y - z| < \varepsilon$. Consequently, (ii) follows from (7). This completes the proof.
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