A HOLOMORPHIC FUNCTION WITH WILD BOUNDARY BEHAVIOR

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To the memory of Darja

Abstract. Let $B$ be the open unit ball in $\mathbb{C}^N$, $N > 1$. It is known that if $f$ is a function holomorphic in $B$, then there are $x \in \partial B$ and an arc $\Lambda$ in $B \cup \{x\}$, with $x$ as one endpoint along which $f$ is constant. We prove

**Theorem.** There exist an $r > 0$ and a function $f$ holomorphic in $B$ with the property that, if $x \in \partial B$ and $\Lambda$ is a path with $x$ as one endpoint, such that $\Lambda - \{x\}$ is contained in the open ball of radius $r$ which is contained in $B$ and tangent to $\partial B$ at $x$, then

$$\lim_{z \in \Lambda, z \to x} f(z)$$

does not exist.

We denote by $B$ the open unit ball in $\mathbb{C}^N$, $N > 1$. For each $x \in \partial B$ and $r$, $0 < r < 1$, let $D(x, r)$ be the open ball of radius $r$, contained in $B$ and tangent to $\partial B$ at $x$. We prove the following

**Theorem.** There exist an $r > 0$ and a function $f$ holomorphic in $B$ such that if $x \in \partial B$ and $\Lambda$ is a path contained in $D(x, r)$, except for its endpoint $x$, then

$$\lim_{z \in \Lambda, z \to x} f(z)$$

does not exist.

It is known that $r$ in the Theorem has to be strictly smaller than 1 [2]; whether or not it can be arbitrarily close to 1 is an open question.

For each $x \in \partial B$ and each $\rho$, $0 < \rho < 1$, let $H(\rho x)$ be the real hyperplane through $\rho x$, tangent to $\rho B$ at $\rho x$. If $R > 0$ let

$$W(\rho x, R) = \{ y \in H(\rho x) : |y - \rho x| < R \}.$$ 

Thus $W(\rho x, R)$ is the relatively open ball in $H(\rho x)$ of radius $R$ centered at $\rho x$.

**Lemma 1.** There is an $r > 0$ with the following property: let $0 < a < 1$; there exists $L \in \mathbb{N}$, numbers $\rho_l$, $1 \leq l \leq L$, $a < \rho_1 < \ldots < \rho_L < \rho_{L+1} = 1$, and numbers $N_l > 0$, $1 \leq l \leq L$, such that $W(x, R_l) \subset \partial(\rho_l B)$ for every $x \in \partial(\rho_l B)$, $1 \leq l \leq L$, and for each $l$, $1 \leq l \leq L$, there is a finite set $T_l \in \partial(\rho_l B)$ such that

(i) $W(x, R_l) \cap H(y) = \emptyset$ whenever $x, y \in T_l, x \neq y, 1 \leq l \leq L$;

(ii) given any $y \in \partial B$ there exist $l$, $1 \leq l \leq L$, and $z \in T_l$ such that if $\Lambda$ is a path joining a point in $\rho_l B$ with $y$, such that $\Lambda - \{y\} \subset D(y, r)$, then $\Lambda$ meets $W(z, R_l)$.
Lemma 2. Let $0 < a < 1$. Let $L, \rho_l, R_l, \text{ and } T_l, 1 \leq l \leq L$, be as in Lemma 1. Given $\epsilon > 0$ and $C < \infty$ there is a polynomial $P$ such that

(i) $\Re P > C \text{ on } \bigcup_{j=1}^{L} \bigcup_{x \in T_j} W(x, R_l);$ 

(ii) $|P| < \epsilon \text{ on } aB.$

Proof. Choose $p_l^j: a < p^j_1 < p^j_2 < \cdots < p^j_L < p^j_{L+1} < 1$ such that for each $l, 1 \leq l \leq L$, $W(x, R_l) \subseteq p^j_{l+1} B$ ($x \in \partial(\rho_l B)$). Fix $l, 1 \leq l \leq L$, and denote $W^j_l = \bigcup_{x \in T_j} W(x, R_l).$ If $\delta_l > 0$ and $C_l < \infty$, then, by Lemma 1(i), one can prove, similarly to the proof of Theorem 4 in [1], that there is a polynomial $P_l$ such that $|P_l| < \delta_l$ on $p^j_l B$ and $\Re P_l > C_l$ on $W^j_l$. If we choose $\delta_l$ and $C_l$ properly, then $P = \sum_{l=1}^{L} P_l$ will have all the required properties. This completes the proof.

Proof of the Theorem. By Lemmas 1 and 2 there exist an $r > 0$, a sequence $a_n, 0 < a_1 < \cdots < 1, \lim a_n = 1,$ and a sequence of sets $W_n, W_n \subseteq a_{n+1} B - a_n B$, such that if $n \in \mathbb{N}, x \in \partial B,$ and $\Lambda$ is a path joining a point in $a_n B$ with $x$, which satisfies $\Lambda - \{x\} \subseteq D(x, r)$, then $\Lambda$ meets $W_n$; moreover, for each $n \in \mathbb{N}, \delta_n > 0,$ and $C_n < \infty$ there is a polynomial $P_n$ such that $|P_n| < \delta_n$ on $a_n B$ and $\Re P_n > C_n$ on $W_n$. If the sequence $C_n$ is chosen inductively to increase to $+\infty$ fast enough, and if the sequence $\delta_n$ is chosen to decrease to 0 fast enough, then the series $\sum_{n=1}^{\infty} P_n$ converges uniformly on compacta in $B$ to a function $f$ holomorphic in $B$ with the property: if $x \in \partial B$ and $\Lambda$ is a path with $x$ as one endpoint which satisfies $\Lambda - \{x\} \subseteq D(x, r)$, then

\[
\limsup_{z \rightarrow x} \Re f(z) = +\infty, \quad \liminf_{z \rightarrow x} \Re f(z) = -\infty.
\]

This completes the proof.

To prove Lemma 1, we first prove three lemmas.

Lemma 3. Let $x, y \in \partial B$ and $|x - y| > 2R/\rho$, where $0 < \rho < 1$ and $R > 0.$ Then $W(\rho x, R) \cap H(\rho y) = \emptyset.$

Proof. Suppose $z \in W(\rho x, R) \cap H(\rho y).$ Then $|z|^2 < \rho^2 + R^2,$ i.e., $z \in W(\rho y, R)$ and, consequently, $\rho|x - y| \leq |z - \rho x| + |z - \rho y| \leq 2R,$ a contradiction.

Lemma 4. Let $0 < r < 1, 0 < \rho < 1,$ and $0 < P < 2^{1/2}.$ Suppose $x, y \in \partial B,$ and $|x - y| < P(1 - \rho)^{1/2}$. Then $x$ and $y$ both lie on the same side of $H(\rho x).$ Moreover, $H(\rho x) \cap D(y, r) \subseteq W(\rho x, Q(1 - \rho)^{1/2}),$ where $Q = (1 - r)P + (2r)^{1/2}.$

Proof. The first statement follows from the fact that $P < 2^{1/2},$ which implies that $|x - y| < (2(1 - \rho))^{1/2}.$ Suppose $H(\rho x) \cap D(y, r)$ is not empty. Write $y = \alpha x + w,$ $\rho < \alpha < 1, |w|^2 + \alpha^2 = 1.$ The center of $D(y, r)$ is at a distance of $|\rho - (1 - r)\alpha|$ from $H(\rho x)$ and at a distance of $(1 - r)(1 - \alpha^2)^{1/2}$ from $R x.$ Consequently, $H(\rho x) \cap D(y, r) \subseteq W(\rho x, R),$, where

\[
R = (1 - r)(1 - \alpha^2)^{1/2} + \left[ r^2 - \left[ \rho - (1 - r)\alpha^2 \right] \right]^{1/2} = (1 - r)^{1/2} + \left[ (1 - \rho) - (1 - r)(1 - \alpha) \right] \cdot \left[ 2r - (1 - \rho) + (1 - r)(1 - \alpha) \right]^{1/2}.
\]
Since $|x - y| < P(1 - \rho)^{1/2}$, we have $(1 - \alpha)^2 + (1 - \alpha^2) < P^2(1 - \rho)$; hence $1 - \alpha < (1 - \rho)^{P^2/2}$, and, consequently,

$$R \leq (1 - r)(1 - \rho)^{1/2}(P/2^{1/2}) \cdot 2^{1/2} + (2r(1 - \rho))^{1/2}.$$ 

This completes the proof.

**Lemma 5.** Let $p \in \mathbb{N}$ and $x \in \partial B$. There exist a neighbourhood $U \subset \partial B$ of $x$, an $r_0 > 0$, and $M \in \mathbb{N}$ such that, for any $r, 0 < r < r_0$, there are finite sets $S_m \subset U$, $1 \leq m \leq M$, such that $U \subset \bigcup_{m=1}^{M} \bigcup_{y \in S_m} (y + rB)$ and $|y - z| \geq pr$ whenever $y, z \in S_m, y \neq z, 1 \leq m \leq M$.

**Proof.** Part 1. We prove the following. Let $W \subset \mathbb{R}^{2N-1}$ be a bounded set and let $k \in \mathbb{N}$. There is a $\mu = \mu(k, N) \in \mathbb{N}$ such that, given any $r > 0$, there are finite sets $T_m \subset \mathbb{R}^{2N-1}$, $1 \leq m \leq \mu$, such that $W \subset \bigcup_{m=1}^{\mu} \bigcup_{y \in T_m} (y + rB)$ (in this part $B$ is the open unit ball in $\mathbb{R}^{2N-1}$) and $|y - z| \geq kr$ whenever $y, z \in T_m, y \neq z, 1 \leq m \leq \mu$.

To do this put $L = 2N - 1$, choose $q \in \mathbb{N}$ such that $q > kL^{1/2}$, and put $\mu = qL$. Let $r > 0$. Define $S \subset \mathbb{R}^L$ by

$$S = \{kr(s_1, s_2, \ldots, s_L) : s_i \in \mathbb{Z}, 1 \leq i \leq L\}.$$ 

Observe that $|y - z| \geq kr$ whenever $y, z \in S, y \neq z$. Further, let $P$ be the set of $\mu$ points in the cube $\{t \in \mathbb{R}^L : 0 \leq t_i \leq kr, 1 \leq i \leq L\}$, defined by

$$P = \{(kr/q)(s_1, s_2, \ldots, s_L) : s_i \in \mathbb{Z}, 1 \leq s_i \leq q, 1 \leq i \leq L\}.$$ 

There are $\mu$ sets of the form $y + S$, where $y \in P$, and they have the following properties:

(a) if $y \in P$ and $w, w \in y + S, w \neq w$, then $|w - w| \geq kr$;

(b) $\mathbb{R}^L = \bigcup_{y \in P} \bigcup_{z \in y + S} (z + K)$,

where $K$ is the cube $\{t \in \mathbb{R}^L : |t_i| < kr/q, 1 \leq i \leq L\}$.

Since $q > kL^{1/2}$, it follows that $kr/q < rL^{-1/2}$; hence $K \subset rB$, which implies $\mathbb{R}^L = \bigcup_{y \in P} \bigcup_{z \in y + S} (z + rB)$. Now the assertion follows from the boundedness of $W$.

Part 2. There exist an open neighbourhood $U' \subset \partial B$ of $x$, an open neighbourhood $V \subset \mathbb{R}^{2N-1}$ of $0$, a constant $c > 0$, and a map $\Psi$ from $V$ onto $U'$ such that

$$(1/c)|u - v| < |\Psi(u) - \Psi(v)| < c|u - v| \quad (u, v \in V).$$

Let $U \subset U'$ be a compact neighbourhood of $x$. The statement of the lemma now follows easily from Part 1. This completes the proof.

**Proof of Lemma 1.** It is enough to prove the following. Let $x \in \partial B$. There exist $M \in \mathbb{N}$, $r > 0$, a neighborhood $U \subset \partial B$ of $x$, and $a, 0 < a < 1$, such that the following holds: Given any $\rho_1, a < \rho_1 < 1$, there exist $R > 0$ and $\rho_m, 1 < m \leq M + 1, \rho_1 < \rho_2 < \cdots < \rho_M < \rho_{M+1} < 1$, such that $\overline{W}(y, r) \subset \rho_{m+1} B$ $(y \in \partial(\rho_m B), 1 \leq m \leq M)$ and, for each $m, 1 \leq m \leq M$, there is a finite set $S_m \subset \partial(\rho_m B)$ such that

(i) $\overline{W}(y, R) \cap H(z) = \emptyset$ whenever $y, z \in S_m, y \neq z, 1 \leq m \leq M$;

(ii) given any $y \in U$ there exist $m, 1 \leq m \leq M$, and $z \in S_m$ such that if $\Delta$ is a path joining a point in $\rho_1 B$ with $y$, where $\Delta = \{y\} \subset D(y, r)$, then $\Delta$ meets $\overline{W}(z, R)$. 

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To prove this let $p = 9$ and let $U$, $r_0$, and $M$ be as in Lemma 5. Choose $P$, $0 < P < 2^{1/2}$, and $r > 0$ such that

$$Q = (1 - r)P + (2r)^{1/2} < (2M)^{-1/2}$$

and

$$8(1 - r) + (2r)^{1/2} / P < p.$$  

Note that, by (2),

$$8Q / P < p.$$  

Choose $a < 1$ so close to 1 that

$$1/2 < a,$$

$$1 - a < r,$$

and

$$P((1 - a)/2)^{1/2} < r_0.$$  

Let $a < \rho_1 < 1$. Set $\delta = (1 - \rho_1)/(2M)$ and let

$$\rho_m = \rho_1 + (m - 1) \delta \quad (1 \leq m \leq M + 1).$$

Put $R = Q(1 - \rho_1)^{1/2}$. By (1) and (4),

$$R < (2M)^{-1/2}(1 - \rho_1)^{1/2} = \delta^{1/2} = (\rho_{m+1} - \rho_m)^{1/2}$$

so $W(y, R) \subseteq \rho_{m+1}B (x \in \partial(\rho_mB), 1 \leq m \leq M)$.

Now let $\varepsilon = P(1 - \rho_1)^{1/2} \cdot 2^{-1/2}$. By (6), $\varepsilon < r_0$. Furthermore, since $(1 - \rho_1)/2 < 1 - \rho_m (1 \leq m \leq M)$, it follows that $\varepsilon < P(1 - \rho_m)^{1/2} (1 \leq m \leq M)$. By (5), $1 - \rho_m < r (1 \leq m \leq M)$; since $P < 2^{1/2}$, it follows, by Lemma 4, that if $y, z \in \partial B, |y - z| < \varepsilon$, then both $y$ and $z$ lie on the same side of $H(\rho_m, y), 1 \leq m \leq M,$ and furthermore,

if $1 \leq m \leq M$ and if $y, z \in \partial B, |y - z| < \varepsilon$, then every path

$$\Lambda,$$

which joins a point in $\rho_mB$ with $z$ and satisfies $\Lambda - \{z\} \subseteq D(z, r)$, meets $W(\rho_m, y, Q(1 - \rho_m)^{1/2}) \subset W(\rho_m, y, R)$.

Furthermore, since $\varepsilon < r_0$, it follows by Lemma 5 that there are finite sets $T_m \subseteq \partial B, 1 \leq m \leq M$, such that

(a) $|y - z| \geq p\varepsilon$ whenever $y, z \in T_m, y \neq z, 1 \leq m \leq M$;

(b) $U \subseteq \bigcup_{m=1}^M \bigcup_{y \in T_m} (y + \varepsilon B)$.

Define $S_m = \rho_mT_m (1 \leq m \leq M)$.

Suppose $1 \leq m \leq M$ and $y, z \in T_m, y \neq z$. By (a) and (3) it follows that

$$|y - z| \geq p\varepsilon > 8(Q/P)\varepsilon = 8(Q/P) \cdot P((1 - \rho_1)/2)^{1/2}$$

$$= 8Q \cdot 2^{-1/2}(1 - \rho_1)^{1/2} > 4R.$$  

By (4), $\rho_m > 1/2$ so, by Lemma 3, $\overline{W(\rho_m, y, R)} \cap H(\rho_m, z) = \emptyset$. This proves (i).

Suppose $y \in U$. By (b) there exist $m, 1 \leq m \leq M,$ and $z \in T_m$ such that $|y - z| < \varepsilon$. Consequently, (ii) follows from (7). This completes the proof.
References


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