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# ON THE KERNEL OF A MARKOV PROJECTION ON $C(X)$

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ABSTRACT. Let $X$ be a compact metric space and $L$ a closed linear subspace of $C(X)$, the real valued continuous functions on $X$. We give necessary and sufficient conditions of an algebraic nature for $L$ to be the kernel of a Markov projection $P$ on $C(X)$. We also characterize compact spaces for which our result holds as those for which the Borsuk-Dugundji simultaneous extension theorem holds.

**1. Introduction.** A projection $P$ on $C(X)$ is *Markov* if $Pe = e$ (where $e$ is the unit function) and $P \geqslant 0$, i.e., $f \geqslant 0$ implies $Pf \geqslant 0$. If $P^*$ is the adjoint of $P$ and $\delta_x$ the Dirac measure at $x$, let $p_x = P^*\delta_x$, so that $p_x$ is a probability measure, and for $f \in C(X)$ we have $Pf(x) = \int f\,dp_x$. Let $\mathbf{P}$ be the set of Borel probability measures on $X$, a compact convex set in $C(X)^*$, relative to the weak*-topology. Then $P^*(\mathbf{P})$ is a compact convex set, and each extreme point $m$ has the form $p_x$ for some $x \in X$—just note that $p^{*-1}(m)$ is a convex compact subset of $\mathbf{P}$, and hence contains an extreme point, which is a $\delta_x$ for some $x \in X$ [**4**, p. 34].

If $m$ is a positive Borel measure, supp $m$ denotes the closed support set of $m$, and if $m$ is any Borel measure, supp $m$ is defined as supp $|m|$. If $P$ is a Markov projection, we define supp $P$ = closure$\bigcup\{\text{supp}\, m\colon P^*m = m\}$. (Note that $m \in \operatorname{ran} P^*$ iff $P^*m = m$.)

The structure of $P$ is pretty well known. Birkhoff [**1**] and Kelley [**3**] characterized those $P$ for which ran $P$ is an algebra by the following properties: for each $x \in X$, $p_x$ is an extreme point of $P^*(\mathbf{P})$, and for each $f \in C(X)$, $Pf$ is constant on supp $p_x$. Moreover, $P$ satisfies the averaging identity $P(fPg) = PfPg$. Lloyd [**5**] showed that if $P$ is an arbitrary Markov projection, then $Pf$ is constant on supp $p_x$ whenever $p_x$ is an extreme point of $P^*(\mathbf{P})$. It follows easily that the natural restriction of $P$ to a projection on $C(\text{supp}\, P)$ satisfies the Birkhoff-Kelley conditions. Later Lloyd and Seever found the following identity for all Markov projection: $P(fPg) = P(PfPg)$ ([**6** and **7**], see also [**9**]).

This formula may be rewritten as $0 = P((f - Pf)Pg)$, i.e., if $f_0 \in \ker P$ and $g_0 \in \operatorname{ran} P$, then $f_0 g_0 \in \ker P$. This condition is not quite strong enough to characterize the kernel of a Markov projection, so we note a natural property of such projections, namely if $f \geqslant 0$, then $Pf = 0$ iff $f$ vanishes on supp $P$. This is an obvious consequence of the fact that for $x \in X$, $p_x$ is a probability measure. Thus, if $P$ is a

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Markov projection we have

(1) $\ker P + \operatorname{ran} P = C(X)$,

(2) $(\operatorname{ran} P)(\ker P) \subset \ker P$,

(3) $I = \{f\colon Pf^2 = 0\}$ is an ideal in $C(X)$.

(Note that if $m$ is a nonpositive Borel measure with $m(e) = 1$, and we define $P$ by $Pf(x) = m(f)$ for all $f \in C(X)$, then (1) and (2) hold, but not (3).)

Our main result is

THEOREM. *Let $X$ be compact metric, $L$ a proper closed linear subspace of $C(X)$, and $M = \{f\colon fL \subset L\}$. If*

(a) $L + M = C(X)$, *and*

(b) $I = \{f\colon f^2 \in L\}$ *is an ideal,*

*then there exists a Markov projection $P$ on $C(X)$ such that $L = \ker P$ and* $\operatorname{ran} P \subset M$.

**2. Preliminaries.** Throughout, $L$ will be a closed subspace of $C(X)$, the real valued continuous functions on $X$, and $M$ and $I$ are as defined in the Theorem. In this section we study the structure of $I$ after we give some definitions.

Let $L^\perp = \{m \in C(X)^*\colon m(f) = 0 \text{ for all } f \in L\}$, and let $(L^\perp)_1$ be the closed unit ball in $L^\perp$, a compact convex set in the weak*-topology. Note that $f \in L$ iff $m(f) = 0$ for all $m \in L^\perp$ (by Hahn-Banach). Obviously, $f \in M$ iff $f\,dm \in L^\perp$ for all $m \in L^\perp$, so $M = \{f \in C(X)\colon fL^\perp \subset L^\perp\}$. Further, $f \in M$ iff $f$ is constant on $\operatorname{supp} m$ for each extreme point $m \in (L^\perp)_1$ [**4**, pp. 35–36]. We also define $Z(I) = \{f^{-1}(0)\colon f \in I\}$ and $\operatorname{supp} L^\perp = \text{closure} \bigcup \{\operatorname{supp} m\colon m \in L^\perp\}$. If $f \in C(X)$ and $A \subset X$, then $f_A$ is the restriction of $f$ to $A$, and $L_A = \{f_A\colon f \in L\}$.

2.1 REMARK. $Z(I) \subset \operatorname{supp} L^\perp$.

PROOF. If $x \notin \operatorname{supp} L^\perp$, then by complete regularity there exists $f \in C(X)$ which vanishes on $\operatorname{supp} L^\perp$, but $f(x) \neq 0$. Then $f^2 \in L$, so $f \in I$ and $x \notin Z(I)$.

2.2 PROPOSITION. *The following are equivalent*:

(a) *$I$ is an ideal,*

(b) $Z(I) = \operatorname{supp} L^\perp$.

PROOF. (b) implies (a). We show $f \in I$ iff $\operatorname{supp} L^\perp \subset f^{-1}(0)$, so that $I$ is the ideal $\{g\colon \operatorname{supp} L^\perp \subset g^{-1}(0)\}$. If $f \in I$, then (b) implies $\operatorname{supp} L^\perp \subset f^{-1}(0)$. If $\operatorname{supp} L^\perp \subset f^{-1}(0)$, then for all $m \in L^\perp$, $0 = m(f^2)$, so $f^2 \in L$ and $f \in I$.

(a) implies (b). To show $\operatorname{supp} L^\perp \subset Z(I)$, let $f \in I$ and $m \in L^\perp$. Let $m = m^+ - m^-$ be the Lebesgue decomposition with $m^+$ supported by the Baire set $A$ and $m^-$ supported by $X \setminus A$. Let $g_n \in C(X)$ with $1 \geqslant g_n \geqslant 0$ and $g_n \to 1_A$ $|m|$-a.e. Now $fg_n \in I$ so $f^2 g_n^2 \in L$, and

$$\int f^2\,dm^+ = \int f^2 1_A\,dm = \lim \int f^2 g_n^2\,dm = 0$$

since $m \in L^\perp$. Likewise $\int f^2\,dm^- = 0$, so $f^2 = 0$ $|m|$-a.e. By continuity, $\operatorname{supp} m \subset f^{-1}(0)$, and since $m$ is arbitrary, $\operatorname{supp} L^\perp \subset f^{-1}(0)$.

2.3 PROPOSITION. *If $M + L = C(X)$ and $m$ is an extreme point of $(L^\perp)_1$, then $m(e) \neq 0$.*

PROOF. Let $S = \operatorname{supp} m$. (Since $L$ is proper, $m \neq 0$.) If $f \in M$ then $f$ is constant on $S$. By hypothesis $C(S) = L_S + M_S$. But then $C(S) = L_S +$ constants, so if $g \in C(S)$ we have $g = h + ce$ with $h \in L_S$ and $c$ constant, whence $m(g) = m(h) + cm(e) = 0 + cm(e)$. If $m(e) = 0$, then $m = 0$, which is impossible.

2.4 PROPOSITION. *If $L + M = C(X)$, then* (a) *and* (b) *in* 2.2 *are equivalent to*
(c) $I \subset M$.

PROOF. (b) implies (c). If $f \in I$, then $f$ is constant (in fact, 0) on $\operatorname{supp} m$ whenever $m \in L^{\perp}$. Hence, $f \in M$ [**4**, pp. 35–36].

(c) implies (b). By 2.1 we always have $Z(I) \subset \operatorname{supp} L^{\perp}$. Conversely, if $f \in I$, then (c) implies $f$ is constant on $\operatorname{supp} m$ whenever $m$ is extreme in $(L^{\perp})_1$. But since $f^2 \in L$ as well, $m(f^2) = 0$. Since $m(e) \neq 0$, $f^2$ must be 0 on $\operatorname{supp} m$. It is an easy consequence of Krein-Milman that sets of the form $\operatorname{supp} m$, with $m$ extreme in $(L^{\perp})_1$, are dense in $\operatorname{supp} L^{\perp}$, so $\operatorname{supp} L^{\perp} \subset f^{-1}(0)$.

2.5 PROPOSITION. *Let $I_0 = \{f \in C(X): f \in L$ and $f^2 \in L\}$. If $I$ is an ideal, then $I = I_0$, and hence $I \subset L \cap M$, provided $L + M = C(X)$.*

PROOF. Clearly, $I_0 \subset I$. If $I$ is an ideal, then $Z(I) = \operatorname{supp} L^{\perp}$, by 2.2, so if $f \in I$, then $0 = m(f) = m(f^2)$ for all $m \in L^{\perp}$, whence $f \in L$ as well as $f^2 \in L$. Thus $f \in I_0$.

2.6 REMARK. Propositions 2.2 and 2.4 remain true if $I$ is replaced by $I_0$. This fact is not needed below, and we omit the easy proof. In §4 we give some examples on the relation between $I$ and $I_0$.

**3. Proof of Theorem.** (i) Let $Z = Z(I)$. By 2.4, hypotheses (a) and (b) of the Theorem imply $Z = \operatorname{supp} L^{\perp}$. We now prove $I = L \cap M$. By 2.5 we already have $I \subset L \cap M$. Conversely, if $f \in L \cap M$, then $f$ is constant on $\operatorname{supp} m$ for $m$ extreme in $(L^{\perp})_1$, while $m(f) = 0$ because $f \in L$. Since by 2.3 $m(e) \neq 0$, we have $f = 0$ on $\operatorname{supp} m$. It follows that $\operatorname{supp} L^{\perp} \subset f^{-1}(0)$, so $f \in I$.

(ii) Since $C(X) = L + M$, $I = L \cap M$, and $Z = Z(I)$, we have $C(Z) = L_Z \oplus M_Z$. Thus, there exists a projection $Q$ on $C(Z)$ whose kernel is $L_Z$ and whose range is $M_Z$. If $e_Z$ is the restriction of $e$ to $Z$, then clearly $Qe_Z = e_Z$, and it remains to show that $Q \geqslant 0$ (and then that $Q$ extends to a Markov projection $P$ on $C(X)$).

(iii) First we show that because (1) $\operatorname{ran}(Q)\ker(Q) \subset \ker(Q)$ and (2) $\operatorname{ran}(Q)$ is an algebra, we have $Q(fQg) = QfQg$ for all $f$ and $g$ in $C(Z)$.

$$\begin{aligned} Q(fQg) &= Q((f - Qf + Qf)Qg) = Q((f - Qf)Qg + Q(QfQg)) \\ &= 0 + QfQg. \end{aligned}$$

(iv) Secondly, if $f \geqslant 0$ and $Qf = 0$, then $f = 0$ on $Z$. Let $F \in C(X)$ satisfy $F \geqslant 0$ and $F_Z = f$. Since $f \in L_Z$, there exists $G \in L$ with $G_Z = f$, i.e., $G_Z = F_Z$. If $m \in L^{\perp}$, then $\operatorname{supp} m \subset Z$, so $m(F) = m(G) = 0$, so $F \in L$. Since $F \geqslant 0$, we have $F^{1/2} \in I \subset M$. Since $M$ is an algebra, $F \in M$, i.e., $F \in L \cap M = I$, so $f = F_Z = 0$.

(v) Finally, suppose there exists $f \in C(Z)$ with $f \geqslant 0$, but $Qf(x) < 0$ for some $x$. The set $V = \{y: Qf(y) < 0\}$ is open in $Z$ relative to the topology generated by the

subalgebra $M_Z = Q(C(Z))$, which is completely regular, but not Hausdorff. Hence, there exists $g \in M_Z$ such that $g(x) = 1$, $g = 0$ off $V$, and $0 \leqslant g \leqslant 1$. Let $h = gf$. Then $h \geqslant 0$, and, by (iii), $Qh = Q(gf) = Q((Qg)f) = QgQf = gQf$. So $Qh(x) = Qf(x) < 0$, $Qh \leqslant 0$ on $V$, and $Qh = 0$ off $V$. Let $k = h - Qh \geqslant 0$. Then $Qk = 0$, so, by (iv), $k = 0$ on $Z$, i.e., $h = Qh$. But this is impossible since $h(x) \geqslant 0$ and $Qh(x) < 0$. (The last three lines were inspired by a homework paper of graduate student Pengyuan Chen.)

(vi) We now show that $Q$ extends to a Markov projection on $C(X)$. Since $X$ is compact metric (and this is the only time metrizabilty is used) there exists a simultaneous extender, i.e., a positive linear map $E$: $C(Z) \to C(X)$ such that, for $x \in Z$, $f(x) = Ef(x)$, and also $Ee_Z = e_X = e$. (See the Borsuk-Dugundji theorem in [**8**, p. 365].) We define $P$ by $Pf(x) = E(Q(f_Z))(x)$. It is easy to check that $P$ is a Markov projection, and we must show that $L = \ker P$ and $\operatorname{ran} P \subset M$.

(vii) To show $L \subset \ker P$, if $f \in L$, then $f_Z \in L_Z$, so $Pf = E(Q(f_Z)) = E(0) = 0$. To show $\ker P \subset L$, suppose $0 = Pf = E(Q(f_Z))$. If $m \in C(X)^*$ and $\operatorname{supp} m \subset Z$, let $m_Z$ be $m$ considered as an element of $C(Z)^*$, so for $g \in C(X)$, $m(g) = m_Z(g_Z)$, and for $g \in C(Z)$, $m_Z(g) = m(Eg)$. Then $m \in L^\perp$ iff $m_Z \in (L_Z)^\perp$. Since $L_Z = \ker Q$ and $Q$ is a projection, $(L_Z)^\perp = \operatorname{ran}(Q^*)$, so $m \in L^\perp$ iff $Q^*m_Z = m_Z$. Hence, for all $m \in L^\perp$,

$$\begin{aligned} m(f) &= m_Z(f_Z) = Q^*m_Z(f_Z) = m_Z(Q(f_Z)) = m(E(Qf_Z)) \\ &= m(Pf) = m(0) = 0. \end{aligned}$$

It follows that $f \in L$.

(viii) To show $\operatorname{ran} P \subset M$, since $L = \ker P$ and $P$ is a Markov operator, property (2) of the introduction says $(\operatorname{ran} P)L \subset L$.

**4. Examples.** We assumed metrizability of $X$ only in order to invoke the Borsuk-Dugundji extension theorem. The following rather surprising result shows that the extension theorem is necessary as well as sufficient.

4. 1 PROPOSITION. *If $X$ is a compact Hausdorff space, the following are equivalent*:

(a) *If $Z$ is a closed subset, there exists a Markov extension operator $E$: $C(Z) \to C(X)$.*

(b) *The result of our main theorem holds for $C(X)$.*

PROOF. We already know that (a) implies (b). Conversely, suppose (b) holds. If $Z$ is closed in $X$, let $L = \{f: f_Z = 0\}$ be an ideal. Then $I = L$, so $I$ is an ideal, and $M = C(X)$, so $M + L = C(X)$. By (b) there exists a Markov projection $P$ with $\ker P = L$. Now $\operatorname{ran} P^* = L^\perp = C(Z)^*$, the space of regular Borel measures on $Z$. That is, if $m \in C(X)^*$ and $\operatorname{supp} m \subset Z$, then $P^*m = m$. We define the extension operator $E$ as follows: if $f \in C(Z)$, let $f_1$ be any norm-preserving extension of $f$ to an element of $C(X)$, and let $Ef = Pf_1$. To show $E$ is well defined, suppose $f_2$ is any other extension of $f$ to an element of $C(X)$. If $x \in X$, then $\operatorname{supp} p_x \subset Z$, so $Pf_1(x) = Pf_2(x) = \int f\,dp_x$. To show $E$ is an extension operator, i.e., $(Ef)_Z = f$, let $x \in Z$. Then $P^*\delta_x = \delta_x$, so $Ef(x) = Pf_1(x) = P^*\delta_x(f_1) = f_1(x) = f(x)$. This completes the proof.

REMARK. The extension property fails for $X = \beta N$ and $Z = \beta N \setminus N$ [**8**, p. 375].

4.2 EXAMPLE. We give an example to show that the hypothesis $L + M = C(X)$ is really needed for Propositions 2.3 and 2.4. Let $X = \{1, 2, 3, 4\}$ with the discrete topology, so that $C(X)$ is essentiallay $R^4$. For simplicity we identify $f \in C(X)$ with its values $(a, b, c, d)$. Let

$$L = \{(a, -a, b, b) : a, b \in R\},$$

so $L^\perp$ is the span of the measures whose values at points are $(1, 1, 0, 0)$ and $(0, 0, 1, -1)$. Now $M = \{(a, a, b, b) : a, b \in R\}$ so $M + L \neq C(X)$. $I = I_0 = \{(0, 0, a, a) : a \in R\}$, which is not an ideal. However, $I \subset M$, so 2.4 fails. Further, $m = (0, 0, \frac{1}{2}, -\frac{1}{2})$ is an extreme measure in $(L^\perp)_1$, but $m(e) = 0$, so 2.3 fails.

We now mention without details some other simple examples we have. (i) $L + M = C(X)$, $I_0$ is not an ideal, $I \neq I_0$, $I \not\subset M$; (ii) $L + M = C(X)$, $I_0$ is an ideal, $I$ is not; (iii) $L + M \neq C(X)$, $I_0$ is an ideal, $I$ is not, and $I \not\subset M$.

**5. Remarks.** I do not know whether our result is valid in noncommutative $C^*$-algebras. It is known that for unital JC algebras, the identity $P(PaPb) = P(aPb)$ holds, where multiplication is the Jordan product [**10**, Lemma 1.1].

From [**2**] it is clear that contractive projections are more complicated than Markov projections, and it is not generally true that $(\operatorname{ran} P)(\ker P) \subset \ker P$. In fact, if $f \in C_C(X)$ (the complex continuous functions) and $m$ is extreme in $(L^\perp)_1$, where $L = \ker P$, then on $\operatorname{supp} m$, $Pf$ is a constant times the Radon-Nikodym derivative $d|m|/dm$. (If $P$ is Markov, then $|m| = \pm m$, so $Pf$ is constant on $\operatorname{supp} m$.) It is an easy consequence of this that $(\operatorname{ran} P)(\operatorname{ran} P)^- \subset \operatorname{mult} P$, or, equivalently, the identity $P(Pf(Pg)^-Ph) = P(f(Pg)^-Ph)$—the bar stands for complex conjugation. In fact, this is proved for general $C^*$-algebras in [**11**, Corollary 3].

Finally, in view of Proposition 4.1, it would be interesting to find characterizations —topological or analytic—of compact spaces for which the extension theorem holds. See [**8**] for references.

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