THE SOLUTION SETS OF EXTREMAL PROBLEMS IN $H^1$

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Abstract. Let $u$ be an essentially bounded function on the unit circle $T$. Let $S_u$ denote the subset of the unit sphere of $H^1$ on which the functional $F \mapsto \int_0^{2\pi} \bar{u}(e^{it}) F(e^{it}) \, dt / 2\pi$ attains its norm. A complete description of $S_u$ is given in terms of an inner function $\phi_0$ and an outer function $\psi_0$ in $H^2$ for which $\psi_0$ is an exposed point in the unit ball of $H^1$. An explicit description is given for the kernel of an arbitrary Toeplitz operator on $H^2$. The exposed points in $H^1$ are characterized; an example is given of a strong outer function in $H^1$ which is not exposed.

1. Introduction. Let $u$ be an essentially bounded function on the unit circle $T$, and let $H^p$ denote the usual Hardy spaces on $T$ for $p \geq 1$. If $I_u$ is the functional on $H^1$ defined by

$$ I_u F = \int_0^{2\pi} \bar{u}(e^{it}) F(e^{it}) \, dt / 2\pi, $$

then an old problem (which will be solved in this article) is to parametrize the set $S_u = \{ F \in S : I_u F = \|I_u\| \}$, where $S$ denotes the unit sphere of $H^1$. This problem was solved in [3] by deLeeuw and Rudin for the special case that $u$ is analytically continuable to $\{ z : |z| > R \}$ for some $R < 1$. Nakazi [6] has also given a partial solution. If $S_u$ is nonempty, then $S_u = S_F / \|F\|$ for some $F$ in $S$. If $S_F / \|F\| = \{ F \}$, $F$ is said to be an exposed point of $S$ (it is the unique point of contact that a certain hyperplane makes with the unit ball of $H^1$). A function $G$ in $H^1$ will be called exposed if $S_G / \|G\|$ is a singleton set. It has been conjectured that a function $F$ is exposed if and only if it is a strong outer function, i.e., if and only if $F$ cannot be factored in the form $F(z) = (z-a)^2 G(z)$, where $G$ is in $H^1$ and $a$ belongs to $T$ (see [3 or 5]). An example will be furnished in §6 to show that a strong outer function need not be exposed. A characterization of the exposed points in $H^1$ will also be given, though the problem of finding an “effective” characterization of the exposed points still remains open.

§2 contains the central results of this paper: a concrete analysis of the spaces $M_k = z^k H^2 \cap (h/\bar{h}) \bar{H}^2$ is given, where $h$ denotes an outer function in $H^2$ and the bar denotes complex conjugation. These spaces arise naturally in the study of stationary stochastic processes (see [2]). Most of the other results in this paper follow from Theorem 3.

In §3 it is shown that the kernel of an arbitrary Toeplitz operator can be expressed as the $L^2$ closure of $L^2 \cap g(H^2 \ominus bH^2)$, where $g^2$ is exposed in $H^1$ and $b$ is an inner function.

§4 translates the preceding result into the language of Hankel operators.
§5 contains a parametrization for $S_u$ and also sheds a little light on the structure of such sets.

§6 contains some remarks about exposed points of $H^1$. In particular, it is shown that strong outer functions need not be exposed, and a characterization of exposed points is given.

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2. The spaces $M_k$. Let $h$ be an outer function in $H^2$. Define $M_k$ as before and let $M'_k$ denote $z^k M_k$. It was shown in [2] that $\dim(M_k/M_{k+1}) \leq 1$ and $M_k = M_{k+1}$ if and only if $M_k = \{0\}$. The following theorem was also proven (using a slightly different definition of $M_k$).

**Theorem 1.** Let $h$ be outer in $H^2$. Then the following are equivalent:

1. $M_k \neq \{0\} = M_{k+1}$.
2. $|h|^2 = |P|^2 |g|^2$, where $g^2$ is exposed in $H^1$ and $P$ is a polynomial of degree $k$ with all of its roots on $T$.
3. $h/h = z^k g^k/g$, where $g^2$ is exposed in $H^1$.
4. $h/h = z^k b g^k/g$, where $g^2$ is exposed in $H^1$ and $b$ is a Blaschke product of order $k$.

It was also noted in [2] that $h^2$ fails to be exposed in $H^1$ if and only if $h/h = zbg/g$ for some inner function $b$ and $g$ outer in $H^2$.

Now, for each $k$ such that $M_k \neq \{0\}$, choose $r_k$ in the unit sphere of $M_k$ to maximize the functional $r \mapsto \Re(r, zk)$, where $(\cdot, \cdot)$ denotes the usual inner product on $L^2(T, d\theta/2\pi)$. Thus, $r_k$ is a real scalar multiple of the projection of $z^k$ on $M_k$.

A moment’s thought reveals that $r_k = z^k g_k$, where $g_k$ is an outer function in $M'_k$. Note that $z^k g_k = (h/h)b kg_k$, where $b_k$ is some inner function. We then have the following result.

**Theorem 2.** If $M_k \neq \{0\}$, each of the following is true:

1. $z^k g_k \in M_k \cap M_{k+1}$.
2. $b_k g_k \in M'_k \cap M'_{k+1}$.
3. $g^2_k$ is exposed in $H^1$.

**Proof.** First, note that if $z^{k+1} f \in M_{k+1}$ and $P_{M_k}$ denotes orthogonal projection on $M_k$, then

$$\langle z^k g_k, z^{k+1} f \rangle = g_k(0)^{-1} \langle P_{M_k} z^k, z^{k+1} f \rangle = g_k(0)^{-1} \langle z^k, z^{k+1} f \rangle = 0.$$ 

Thus, (2.5) holds. Item (2.6) follows from (2.5) by noting that $b_k g_k = (h/h)z^k g_k$.

Now assume (2.7) fails. Then we may write $g_k/g_k = zbf/f$, where $b$ is inner and $f$ is outer in $H^2$. Replacing $f$ by $(1+b)f$, we may assume, without loss of generality, that $g_k/g_k = zf/f$, where $f$ is outer. Thus $z^{k+1} f \in M_{k+1}$. Now $\langle g_k, f \rangle = \langle zf, g_k \rangle$ and both quantities equal zero by (2.5). Set

$$s_t = z^k g_k + tz^{k+1} f(0) f,$$

where $t$ is a real parameter. Then $s_t \in M_k$ and

$$\frac{\langle s_t, z^k \rangle}{\|s_t\|_2} = \frac{\langle r_k, z^k \rangle + t|f(0)|^2}{[1 + t^2|f(0)|^2\|f\|_2^2]^{1/2}}.$$
For small positive $t$ the extremal property of $r_k$ is contradicted by (2.8), so we must have $g_k^2$ exposed in $H^1$, and Theorem 2 is proved.

It turns out that $b_k$ and $g_k$ can be defined recursively from $b_0$ and $g_0$. Let $a_k = b_k(0)$.

**Theorem 3.** For each $k$ such that $M_{k+1} \neq \{0\}$, there exists a $\lambda_{k+1} \in \mathbb{C}$ such that

1. $g_{k+1} = \lambda_{k+1} g_k \cdot (1 - \bar{a}_k b_k)$,
2. $b_{k+1} = (b_k - a_k)/z(1 - \bar{a}_k b_k)$.

**Proof.** Write $\psi_{k+1} = (b_k - a_k)/z(1 - \bar{a}_k b_k)$, and $f_{k+1} = g_k \cdot (1 - \bar{a}_k b_k)$. Then $\psi_{k+1}$ is an inner function and $f_{k+1}$ is outer. Using the relation $h/h = z b_k g_k/g_k$, we have

$$z^{k+1} \psi_{k+1} f_{k+1} = z^k b_k - a_k - a_k b_k \cdot \frac{g_k}{1 - \bar{a}_k b_k} \cdot \frac{1 - \bar{a}_k b_k}{b_k}$$

Hence, $z^{k+1} f_{k+1} \in M_{k+1}$. Now, let $\phi = h/h$. Then for $n > k + 1$,

$$\langle z^{k+1} f_{k+1}, z^n g_n \rangle = \langle z^{k+1} g_k, z^n g_n \rangle - \bar{a}_k \langle z^{k+1} b_k g_k, z^n g_n \rangle$$

$$= \langle z^k g_k, z^{n-1} g_n \rangle - \bar{a}_k \langle z^{k+1} b_k g_k, z^n g_n \rangle$$

$$= 0 - \bar{a}_k \langle z^k \phi g_k, \bar{b}_n g_n \phi \rangle = -\bar{a}_k \langle z^{k+1} b_n g_n, z^k g_k \rangle.$$

Now, $z^{k+1} b_n g_n \in M_{k+1}$, so this last inner product vanishes. By Theorem 2 we then have $z^{k+1} f_{k+1} = \lambda_{k+1} z^{k+1} g_k$ for some $\lambda_{k+1} \in \mathbb{C}$. Now, it is easily checked that $b_{k+1} = \psi_{k+1}$.

**Corollary 4.** $L^2 \cap g_0(H^2 \ominus z b_0 H^2)$ is a dense subset of $M_0$.

**Proof.** Fix $h, b_k, g_k$ and $a_k$ as above. Let $h_0 = 1 + b_0$, $m_k = z^k H^2 \ominus (h_0/\bar{h}_0) H^2$, and $m'_k = \bar{z}^k m_k$. Carrying out the program of Theorem 2, let $\{z^k G_k\}$ be the o.n. basis for $m_0$ with $z^k G_k \in m_k \ominus m_{k+1}$. Then $G_0 = 1$ and $b_0 = h_0/\bar{h}_0 = b_0 G_0/\bar{G}_0$, and for $k \geq 1$, $h_0/\bar{h}_0 = z^k B_k G_k/\bar{G}_k$, where $B_k$ is inner. The proof of Proposition 4 shows that $B_k = b_k$ for each $k$. Thus, $\beta = \{1, z(1 - \bar{a}_0 b_0), z^2(1 - \bar{a}_1 b_1), \ldots\}$ forms an orthogonal basis for $m_0 = H^2 \ominus b_0 H^2 = H \ominus z b_0 H^2$. But $g_0 \beta = \{g_0, z g_0(1 - \bar{a}_0 b_0), \ldots\}$, which is a spanning set for $M_0$. Also, if an $L^2$ function $g$ belongs to $g_0(H^2 \ominus b_0 H^2)$, it clearly belongs to $M_0$. Thus $M_0$ is the $L^2$ closure of $L^2 \cap g_0(H^2 \ominus z b_0 H^2)$.

**Remark.** It is easily seen that $M_0$ is finite dimensional if and only if $b_0$ is a finite Blaschke product, in which case the dimension of $M_0$ equals one plus the order of $b_0$. The construction of the $g_k$ provides a simple way to obtain an orthogonal basis for $H^2 \ominus b H^2$ in the case where $b$ is an inner function which is not a Blaschke product.

3. **The kernels of Toeplitz operators.** Let $f$ be an essentially bounded function on $T$ and let $T_f$ denote the Toeplitz operator on $H^2$ defined by $T_f g = P(fg)$, where $P$ denotes orthogonal projection from $L^2$ onto $H^2$. If $f = \bar{b}$, where $b$ is an inner function, then $\text{Ker} T_f$, the kernel of $T_f$, is just $H^2 \ominus b H^2$ and projection onto this space is easily carried out. It turns out that, in general, $\text{Ker} T_f$ is a weighted version of the above. This was shown independently by Nakazi [6] for the case that $\text{Ker} T_f$ is finite dimensional.
LEMMA 5. Let $f \in L^\infty$ and suppose $\text{Ker}(T_f) \neq \{0\}$. Then there exists an outer function $h$ in $H^2$ such that $\text{Ker}(T_f) = \text{Ker}(T_{\frac{h}{h}})$.

PROOF. Let $g$ be a nontrivial function in $\text{Ker}(T_f)$. Then there exists a $k \in H^2$ such that $fg = \overline{z}k$ ($z$ denotes the identity function of $T$). Thus, $|f| = |k/g|$, so $\log |f|$ is integrable, hence we may write $f = uF$, where $F$ is outer in $H^\infty$ and $|u| = 1$ a.e. on $T$. Now, if $g \in H^2$, write $g = BG$, where $G$ is outer and $B$ is inner. Then $T_f g = 0$ if and only if $u(F/\overline{F})g$ is in $(H^2)^\perp$, i.e., if and only if $T_{uF/\overline{F}} g = 0$. Note also that

$$uF/\overline{F} = zB \overline{B_1}\overline{G}/G = (1 + zBB_1)G/(1 + zBB_1)G.$$  

Now, let $h = (1 + zBB_1)G$. The first factor takes its values in the right half-plane and is bounded, hence it is outer in $H^\infty$. Thus, $h$ is seen to be outer in $H^2$ with

$$\text{Ker}(T_f) = \text{Ker}(T_{uF/\overline{F}}) = \text{Ker}(T_{\frac{h}{h}}).$$

This proves the lemma.

THEOREM 6. Suppose $f$ is not identically zero and $\text{Ker} T_f$ is nontrivial. Then there is an outer function $h$, an inner function $b_1$, and an outer function $g_1$ whose square is exposed in $H^1$ such that $L^2 \cap g_1(H^2 \ominus zb_1 H^2)$ is dense in $\text{Ker} T_f$. In fact, an orthogonal basis for $\text{Ker} T_f$ is given by

$$\{g_1, z(1 - \overline{a_1}b_1)g_1, z^2(1 - \overline{a_1}b_1)(1 - \overline{a_2}b_2)g_1, \ldots\},$$

where $a_k, b_k$, and $g_k$ are related to $h$ as in §2.

PROOF. From the previous lemma we may assume that $\text{Ker} T_f = \text{Ker} T_{\frac{h}{h}}$, where $h$ is outer in $H^2$. It is easily checked that this last kernel is $H^2 \cap \overline{z(h/h)}H^2 = M'$. This is spanned by the set $\{g_1, zg_2, z^2g_3, \ldots\}$, which, by Theorem 3, spans the same space as does

$$g_1 \{z(1 - \overline{a_1}b_1), z^2(1 - \overline{a_1}b_1)(1 - \overline{a_2}b_2), \ldots\}.$$  

This is contained in $L^2 \cap g_1(H^2 \ominus zb_1 H^2)$ which, in turn, is a subset of $M'$.

4. Hankel operators. For an essentially bounded function $f$ on $T$, let $H_f$ denote the Hankel operator from $H^2$ into $(H^2)^\perp$ defined by $H_f g = (I - P)(fg)$, where $I$ is the identity operator on $L^2$. Then by a theorem of Nehari,

$$\|H_f\| = \inf\{\|f - g\|_\infty : g \in H^\infty\}$$

(see [7]). Using a normal families argument, we may assume, without loss of generality, that $\|H_f\| = \|f\|_\infty$. Let $N = \{g \in H^2 : \|H_f g\|_2 = \|H_f\|_2 \cdot \|g\|_2\}$. If $N$ contains a nontrivial function $g$, then we have

$$\|f\|_\infty \|g\|_2 = \|H_f g\|_2 \leq \|f\|_2 \|g\|_2 \leq \|f\|_\infty \|g\|_2,$$

so $|f| = \|f\|_\infty$ a.e. on $T$. To describe $N$ in the case that $N \neq \{0\}$, we then assume, without loss of generality, that $|f| = 1$ a.e. on $T$. Then

$$N = \text{Ker}(I - H_f^* H_f)^{1/2} = \text{Ker}(T_f^* T_f)^{1/2} = \text{Ker}(T_f^* T_f) = \text{Ker} T_f.$$

By Lemma 5 we may write $f = h/\overline{h}$ for some outer function $h$ in $H^2$, so $N$ is of the form in Theorem 6.
5. A parametrization of $S_u$. Let $S_u$ be defined as in §1. If $S_u$ is nonempty, then $S_u = S_{F/|F|}$ for some function $F$ in $H^1$. As before, we may assume, without loss of generality, that $F$ is outer. Let $h = F^{1/2}$. Now form the spaces $M_k$ as in §2 along with $b_k$ and $g_k$.

**Theorem 7.** Let $S_u$ be nonempty. If $h, b_k, g_k$ are as above, then every $G$ in $S_u$ is of the form $G = (h/\bar{h})|g|^2$, where $||g||_2 = 1$ and $g$ is in the $L^2$ closure of $L^2 \cap g_0(H^2 \oplus z b_0 H^2)$.

**Proof.** We have $F/|F| = h/\bar{h} = b_0 g_0 / \bar{g}_0$ in the notation of §2. If $G \in S_{F/|F|}$, then write $G = Bg^2$, where $g$ is outer in $H^2$ and $B$ is inner. Then $G/|G| = Bg/\bar{g} = h/\bar{h}$, so $Bg$ is seen to be in $H^2 \cap (h/\bar{h})H^2 = M_0$. Thus,

$$G = (h/\bar{h})|G| = (h/\bar{h})|Bg|^2.$$ 

Conversely, if $Bg \in M_0$, where $B$ is inner and $g$ is outer with unit norm in $H^2$, then $Bg = (h/\bar{h})B_1 \bar{g}$, where $B_1$ is inner, so $(h/\bar{h})|Bg|^2 = BB_1 g^2$ and

$$\text{Arg}(BB_1 g^2) = \text{Arg}(h/\bar{h}) = \text{Arg}(F/|F|),$$

so $(h/\bar{h})|Bg|^2$ belongs to $S_{F/|F|}$. The theorem now follows from Corollary 4.

It was noted in [2] that if $S_{F/|F|}$ contains a strong outer function and is not a singleton, then it contains functions with arbitrarily many zeroes in the unit disk $D$. This is a direct consequence of Theorem 1. It was also conjectured that, in this case, $S_{F/|F|}$ must contain a function whose inner part is not a finite Blaschke product. Corollary 4 provides an affirmative answer to this conjecture. To see this, note that if there is no bound for the number of zeroes for functions in $S_{F/|F|}$, then the related space $M_0$ must be infinite dimensional. Hence $b_0$ is not a finite Blaschke product, yet $b_0 g_0^2$ is in $S_{F/|F|}$.

6. Exposed points in $H^1$. It was noted by deLeuuw and Rudin in [3] that every exposed point $F$ in $H^1$ is a strong outer function. It was conjectured by Nakazi [6] that the converse also holds. Theorem 1 provides some evidence of this. Other sufficient conditions for $F$ to be exposed are discussed in [2]. An easy way to construct an outer function in $H^1$ which is not exposed is to take $F = (1 + B)^2$, where $B$ is an inner function. $F$ is outer since $1 + B$ takes its values in the right half-plane, and $F/|F| = B$, so $F$ is not exposed. Consider a Blaschke product $B$ with zero sequence $\{w_n\}$. It was shown by Ahern and Clark in [1] that $1 + B = (z - a)g$ for some $g$ in $H^2$ and $a \in T$ only if

$$\sum_{n=1}^{\infty} \frac{1 - |w_n|^2}{1 - \bar{a}w_n|^2} < \infty. \tag{6.1}$$

However, there is an example (cited in [1]) due to Frostman [4] of a Blaschke product for which the sum in (6.1) diverges for every $a \in T$. It then follows that for this choice of $B$, $(1 + B)^2$ is a strong outer function which is not exposed. Hence the conjecture is false.

It would be of interest to have a usable characterization of the exposed points of $H^1$. The next theorem gives a characterization, though its usability is questionable.
THEOREM 8. Let $h$ be an outer function in the unit sphere of $H^2$. Then $h^2$ is not exposed if and only if there exists a positive constant $C < 1$ such that

$$|h(0)| \leq C\|T_{h/\overline{h}}(1 + zf)\|_2$$

for every $f \in H^2$.

**Proof.** Using the notation of §2, note that $h \in M_0$ and $h^2$ is exposed if and only if $M_1 = M'_1 = \{0\}$. Now, $M'_1 = \text{Ker} T_{h/\overline{h}} = (\text{Range} T_{h/\overline{h}})^\perp$. (See [7] for a discussion of Toeplitz operators.) Thus, if $M'_1 = \{0\}$, $h \in (M'_1)^\perp$, so $h$ is in the closure of the range of $T_{h/\overline{h}}$. In this case,

$$1 = \|h\|_2 = \sup \{|\langle h, T_{h/\overline{h}} g \rangle| / \|T_{h/\overline{h}} g\|_2\} = \sup \{|\langle h, (h/\overline{h}) g \rangle| / \|T_{h/\overline{h}} g\|_2\} = \sup \{|h(0)| ; |g(0)| / \|T_{h/\overline{h}} g\|_2\},$$

where the supremum is taken over all $g \in H^2$ such that $T_{h/\overline{h}} g \neq 0$. Note that

$$\langle h, T_{h/\overline{h}} g \rangle = \overline{h(0)} \cdot g(0),$$

so if $g(0) \neq 0$, then $T_{h/\overline{h}} g \neq 0$. Hence, (6.2) can hold only if $M'_1 \neq \{0\}$.

Suppose, conversely, that (6.2) fails to hold. Then $h$ is orthogonal to $M'_1$ so, by Theorem 2, $h$ is a scalar multiple of $b_0g_0$. But since $h$ is outer, we must have that $b_0$ is a unimodular constant, and hence, by the same theorem, $h^2$ must be an exposed point of $H^1$. This establishes the theorem.

Finally, it seems that the key to characterizing the exposed points of $H^1$ lies in understanding the behavior of bounded outer functions in the left invariant subspaces $H^2 \ominus bH^2$ for arbitrary inner functions $b$. For, suppose that $F$ is outer in $S$ and is not exposed. Take any other outer function $G$ in $S_{F/|F|}$. We then have $(F + G)/2 = BK \in S_{F/|F|}$, where $B$ is a nonconstant inner function and $K$ is outer (this is because $(F + G)/2$ is not an extreme point of $S$: see [3]). Note that $|K| \geq |G|/2$ and $|K| \geq |F|/2$ a.e. on $T$ since $F$ and $G$ have the same argument a.e. Thus, $F = 2K(B - G/2K)$. Now $\text{Arg}(G/2K) = \text{Arg}(B)$ a.e., and $|G/2K| < 1$ a.e., and $B - G/2K$ is outer since $F$ is. Note further that $|B - G/2K| = 1 - |G/2K|$. Thus we may write $F = 2Kg^2$, where $g$ is outer in $H^2 \ominus zBH^2$ and $0 < |g| < 1$ a.e. on $T$. Of course, when $B$ turns out to be a finite Blaschke product, $g^2$ has double zeroes on $T$ by Theorem 1.

**Added in Proof.** The author can now show that $M_0 = g_0(H^2 \ominus zB_0H^2)$.

**References**


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