THE SOLUTION SETS OF EXTREMAL PROBLEMS IN $H^1$

ERIC HAYASHI

ABSTRACT. Let $u$ be an essentially bounded function on the unit circle $T$. Let $S_u$ denote the subset of the unit sphere of $H^1$ on which the functional $F \mapsto \int_0^{2\pi} \overline{u(e^{it})}F(e^{it})\,dt/2\pi$ attains its norm. A complete description of $S_u$ is given in terms of an inner function $\beta_0$ and an outer function $\gamma_0$ in $H^2$ for which $g_0$ is an exposed point in the unit ball of $H^1$. An explicit description is given for the kernel of an arbitrary Toeplitz operator on $H^2$. The exposed points in $H^1$ are characterized; an example is given of a strong outer function in $H^1$ which is not exposed.

1. Introduction. Let $u$ be an essentially bounded function on the unit circle $T$, and let $H^p$ denote the usual Hardy spaces on $T$ for $p \geq 1$. If $I_u$ is the functional on $H^1$ defined by

$$I_u F = \int_0^{2\pi} \overline{u(e^{i\theta})} F(e^{i\theta}) \,d\theta/2\pi,$$

then an old problem (which will be solved in this article) is to parametrize the set $S_u = \{ F \in S : I_u F = \| I_u \| \}$, where $S$ denotes the unit sphere of $H^1$. This problem was solved in [3] by deLeeuw and Rudin for the special case that $u$ is analytically continuable to $\{ z : |z| > R \}$ for some $R < 1$. Nakazi [6] has also given a partial solution. If $S_u$ is nonempty, then $S_u = S_F/|F|$ for some $F$ in $S$. If $S_F/|F| = \{ F \}$, $F$ is said to be an exposed point of $S$ (it is the unique point of contact that a certain hyperplane makes with the unit ball of $H^1$). A function $G$ in $H^1$ will be called exposed if $S_{G/|G|}$ is a singleton set. It has been conjectured that a function $F$ is exposed if and only if it is a strong outer function, i.e., if and only if $F$ cannot be factored in the form $F(z) = (z-a)^2 G(z)$, where $G$ is in $H^1$ and $a$ belongs to $T$ (see [3 or 5]). An example will be furnished in §6 to show that a strong outer function need not be exposed. A characterization of the exposed points in $H^1$ will also be given, though the problem of finding an “effective” characterization of the exposed points still remains open.

§2 contains the central results of this paper: a concrete analysis of the spaces $M_k = z^k H^2 \cap (h/h)\overline{H^2}$ is given, where $h$ denotes an outer function in $H^2$ and the bar denotes complex conjugation. These spaces arise naturally in the study of stationary stochastic processes (see [2]). Most of the other results in this paper follow from Theorem 3.

In §3 it is shown that the kernel of an arbitrary Toeplitz operator can be expressed as the $L^2$ closure of $L^2 \cap g(H^2 \ominus bH^2)$, where $g^2$ is exposed in $H^1$ and $b$ is an inner function.

§4 translates the preceding result into the language of Hankel operators.

1980 Mathematics Subject Classification. Primary 30D55, 60G10, 60G25, 42A10, 47B35.

©1985 American Mathematical Society

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
§5 contains a parametrization for $S_u$ and also sheds a little light on the structure of such sets.

§6 contains some remarks about exposed points of $H^1$. In particular, it is shown that strong outer functions need not be exposed, and a characterization of exposed points is given.

The author would like to thank Nicholas P. Jewell for introducing him to this area of study and Donald Sarason for stimulating correspondence and conversation on the subject.

2. The spaces $M_k$. Let $h$ be an outer function in $H^2$. Define $M_k$ as before and let $M'_k$ denote $z^k M_k$. It was shown in [2] that $\dim(M_k/M_{k+1}) \leq 1$ and $M_k = M_{k+1}$ if and only if $M_k = \{0\}$. The following theorem was also proven (using a slightly different definition of $M_k$).

**Theorem 1.** Let $h$ be outer in $H^2$. Then the following are equivalent:

1. $M_k \neq \{0\} = M_{k+1}$.
2. $|h|^2 = |P|^2 |g|^2$, where $g^2$ is exposed in $H^1$ and $P$ is a polynomial of degree $k$ with all of its roots on $T$.
3. $h/h = zkg\bar{g}$, where $g^2$ is exposed in $H^1$.
4. $h/h = bg/g$, where $g^2$ is exposed in $H^1$ and $b$ is a Blaschke product of order $k$.

It was also noted in [2] that $h^2$ fails to be exposed in $H^1$ if and only if $h/h = zbg\bar{g}$ for some inner function $b$ and $g$ outer in $H^2$.

Now, for each $k$ such that $M_k \neq \{0\}$, choose $r_k$ in the unit sphere of $M_k$ to maximize the functional $r \mapsto \Re(r, z^k)$, where $(\cdot, \cdot)$ denotes the usual inner product on $L^2(T, d\theta/2\pi)$. Thus, $r_k$ is a real scalar multiple of the projection of $z^k$ on $M_k$.

A moment’s thought reveals that $r_k = zkg_k$, where $g_k$ is an outer function in $M'_k$. Note that $z^k g_k = (h/h) bkg_k$, where $b_k$ is some inner function. We then have the following result.

**Theorem 2.** If $M_k \neq \{0\}$, each of the following is true:

1. $z^k g_k \in M_k \oplus M_{k+1}$.
2. $b_k g_k \in M'_k \oplus M'_{k+1}$.
3. $g^2_k$ is exposed in $H^1$.

**Proof.** First, note that if $z^{k+1} f \in M_{k+1}$ and $P_{M_k}$ denotes orthogonal projection on $M_k$, then

$$\langle z^k g_k, z^{k+1} f \rangle = g_k(0)^{-1} \langle P_{M_k} z^k, z^{k+1} f \rangle = g_k(0)^{-1} \langle z^k, z^{k+1} f \rangle = 0.$$ 

Thus, (2.5) holds. Item (2.6) follows from (2.5) by noting that $b_k g_k = (h/h) z^k g_k$.

Now assume (2.7) fails. Then we may write $g_k/\bar{g}_k = zbf/f$, where $b$ is inner and $f$ is outer in $H^2$. Replacing $f$ by $(1+b)f$, we may assume, without loss of generality, that $g_k/\bar{g}_k = zf/f$, where $f$ is outer. Thus $z^{k+1} f \in M_{k+1}$. Now $\langle g_k, f \rangle = \langle zf, g_k \rangle$ and both quantities equal zero by (2.5). Set

$$s_t = z^k g_k + tz^k f(0)f,$$

where $t$ is a real parameter. Then $s_t \in M_k$ and

$$\frac{\langle s_t, z^k \rangle}{\|s_t\|_2} = \frac{\langle r_k, z^k \rangle + t|f(0)|^2}{[1 + t^2|f(0)|^2\|f\|_2^2]^{1/2}}.$$
For small positive \( t \) the extremal property of \( r_k \) is contradicted by (2.8), so we must have \( g_k^2 \) exposed in \( H^1 \), and Theorem 2 is proved.

It turns out that \( b_k \) and \( g_k \) can be defined recursively from \( b_0 \) and \( g_0 \). Let \( a_k = b_k(0) \).

**Theorem 3.** For each \( k \) such that \( M_{k+1} \neq \{0\} \), there exists a \( \lambda_{k+1} \in \mathbb{C} \) such that

\[
\begin{align*}
(1) & \quad g_{k+1} = \lambda_{k+1} g_k \cdot (1 - \overline{a}_k b_k), \\
(2) & \quad b_{k+1} = (b_k - a_k)/z(1 - \overline{a}_k b_k).
\end{align*}
\]

**Proof.** Write \( \psi_{k+1} = (b_k - a_k)/z(1 - \overline{a}_k b_k) \), and \( f_{k+1} = g_k \cdot (1 - \overline{a}_k b_k) \). Then \( \psi_{k+1} \) is an inner function and \( f_{k+1} \) is outer. Using the relation \( h/h = z^k b_k g_k / \overline{g_k} \), we have

\[
\frac{z^{k+1} \psi_{k+1} f_{k+1}}{f_{k+1}} = \frac{z^k b_k - a_k}{1 - \overline{a}_k b_k} \cdot \frac{g_k}{\overline{g_k}} \cdot \frac{1 - \overline{a}_k b_k}{1 - a_k b_k} \cdot \frac{b_k}{b_k} = z^k b_k g_k / \overline{g_k} = h/h.
\]

Hence, \( z^{k+1} f_{k+1} \in M_{k+1} \). Now, let \( \phi = h/h \). Then for \( n > k + 1 \),

\[
\begin{align*}
\langle z^{k+1} f_{k+1}, z^n g_n \rangle &= \langle z^{k+1} g_k, z^n g_n \rangle - \overline{\alpha}_k \langle z^{k+1} b_k g_k, z^n g_n \rangle \\
&= \langle z^k g_k, z^{n-1} g_n \rangle - \overline{\alpha}_k \langle z^{k+1} b_k g_k, z^n g_n \rangle \\
&= 0 - \overline{\alpha}_k \langle z^{k+1} b_n g_n, z^k g_k \rangle.
\end{align*}
\]

Now, \( z^{k+1} b_n g_n \in M_{k+1} \), so this last inner product vanishes. By Theorem 2 we then have \( z^{k+1} f_{k+1} = \lambda_{k+1} z^{k+1} g_k \) for some \( \lambda_{k+1} \in \mathbb{C} \). Now, it is easily checked that \( b_{k+1} = \psi_{k+1} \).

**Corollary 4.** \( L^2 \cap g_0(H^2 \oplus z b_0 H^2) \) is a dense subset of \( M_0 \).

**Proof.** Fix \( h, b_k, g_k \) and \( a_k \) as above. Let \( h_0 = 1 + b_0 \), \( m_k = z^k H^2 \cap (h_0 / \overline{h_0}) H^2 \), and \( m'_{k} = z^k m_k \). Carrying out the program of Theorem 2, let \( \{z^k G_k \} \) be the o.n. basis for \( m_0 \) with \( z^k G_k \in m_k \oplus m_{k+1} \). Then \( G_0 = 1 \) and \( b_0 = h_0 / \overline{h_0} = b_0 G_0 / \overline{G_0} \), and for \( k \geq 1 \), \( h_0 / \overline{h_0} = z^k B_k G_k / \overline{G_k} \), where \( B_k \) is inner. The proof of Proposition 4 shows that \( B_k = b_k \) for each \( k \). Thus, \( \beta = \{1, z(1 - \overline{a}_0 b_0), z^2(1 - \overline{a}_1 b_1), \ldots \} \) forms an orthogonal basis for \( m_0 = H^2 \cap b_0 H^2 = H \oplus z b_0 H^2 \). But \( g_0 \beta = \{g_0, z g_0 (1 - \overline{a}_0 b_0), \ldots \} \), which is a spanning set for \( M_0 \). Also, if an \( L^2 \) function \( g \) belongs to \( g_0(H^2 \cap b_0 H^2) \), it clearly belongs to \( M_0 \). Thus \( M_0 \) is the \( L^2 \) closure of \( L^2 \cap g_0(H^2 \oplus z b_0 H^2) \).

**Remark.** It is easily seen that \( M_0 \) is finite dimensional if and only if \( b_0 \) is a finite Blaschke product, in which case the dimension of \( M_0 \) equals one plus the order of \( b_0 \). The construction of the \( g_k \) provides a simple way to obtain an orthogonal basis for \( H^2 \oplus b H^2 \) in the case where \( b \) is an inner function which is not a Blaschke product.

**3. The kernels of Toeplitz operators.** Let \( f \) be an essentially bounded function on \( T \) and let \( T_f \) denote the Toeplitz operator on \( H^2 \) defined by \( T_f g = P(f g) \), where \( P \) denotes orthogonal projection from \( L^2 \) onto \( H^2 \). If \( f = \overline{b} \), where \( b \) is an inner function, then \( \text{Ker} T_f \), the kernel of \( T_f \), is just \( H^2 \oplus b H^2 \) and projection onto this space is easily carried out. It turns out that, in general, \( \text{Ker} T_f \) is a weighted version of the above. This was shown independently by Nakazi [6] for the case that \( \text{Ker} T_f \) is finite dimensional.
THE SOLUTION SETS OF EXTREMAL PROBLEMS IN $H^1$

LEMMA 5. Let $f \in L^\infty$ and suppose $\text{Ker}(T_f) \neq \{0\}$. Then there exists an outer function $h$ in $H^2$ such that $\text{Ker}(T_f) = \text{Ker}(T_{h/h})$.

PROOF. Let $g$ be a nontrivial function in $\text{Ker}(T_f)$. Then there exists a $k \in H^2$ such that $fg = \overline{z}k$ ($z$ denotes the identity function of $T$). Thus, $|f| = |k/g|$, so $\log |f|$ is integrable, hence we may write $f = uF$, where $F$ is outer in $H^\infty$ and $|u| = 1$ a.e. on $T$. Now, if $g \in H^2$, write $g = BG$, where $G$ is outer and $B$ is inner. Then $T_f g = 0$ iff $uFBG = \overline{z}B_1 \overline{F}G$ for some inner function $B_1$. Thus, $T_f g = 0$ if and only if $u(F/F)g$ is in $(H^2)_{-1}$, i.e., if and only if $TuF/Fg = 0$. Note also that

$$uF/F = \overline{zB_1} \overline{G}/G = (1 + zBB_1)G/(1 + zBB_1)G.$$ 

Now, let $h = (1 + zBB_1)G$. The first factor takes its values in the right half-plane and is bounded, hence it is outer in $H^\infty$. Thus, $h$ is seen to be outer in $H^2$ with

$$\text{Ker} T_f = \text{Ker} TuF/F = \text{Ker}(T_{h/h}).$$

This proves the lemma.

THEOREM 6. Suppose $f$ is not identically zero and $\text{Ker} T_f$ is nontrivial. Then there is an outer function $h$, an inner function $b_1$, and an outer function $g_1$ whose square is exposed in $H^1$ such that $L^2 \cap g_1(H^2 \ominus zb_1 H^2)$ is dense in $\text{Ker} T_f$. In fact, an orthogonal basis for $\text{Ker} T_f$ is given by

$$\{g_1, z(1 - \overline{a}_1b_1)g_1, z^2(1 - \overline{a}_1b_1)(1 - \overline{a}_2b_2)g_1, \ldots\},$$

where $a_k, b_k$, and $g_k$ are related to $h$ as in §2.

PROOF. From the previous lemma we may assume that $\text{Ker} T_f = \text{Ker}(T_{h/h})$, where $h$ is outer in $H^2$. It is easily checked that this last kernel is $H^2 \cap \overline{z}(h/h)H^2 = M_1'$. This is spanned by the set $\{g_1, zg_2, z^2g_3, \ldots\}$, which, by Theorem 3, spans the same space as does

$$g_1\{1, z(1 - \overline{a}_1b_1), z^2(1 - \overline{a}_1b_1)(1 - \overline{a}_2b_2), \ldots\}.$$ 

This is contained in $L^2 \cap g_1(H^2 \ominus zb_1 H^2)$ which, in turn, is a subset of $M_1'$.

4. Hankel operators. For an essentially bounded function $f$ on $T$, let $H_f$ denote the Hankel operator from $H^2$ into $(H^2)_{-1}$ defined by $H_f g = (I - P)(fg)$, where $I$ is the identity operator on $L^2$. Then by a theorem of Nehari,

$$\|H_f\| = \inf \{\|f - g\|_\infty : g \in H^\infty\}$$

(see [7]). Using a normal families argument, we may assume, without loss of generality, that $\|H_f\| = \|f\|_\infty$. Let $N = \{g \in H^2 : \|H_f g\|_2 = \|H_f\| \cdot \|g\|_2\}$. If $N$ contains a nontrivial function $g$, then we have

$$\|f\|_\infty \|g\|_2 = \|H_f g\|_2 \leq \|f\|_2 \leq \|f\|_\infty \|g\|_2,$$

so $|f| = \|f\|_\infty$ a.e. on $T$. To describe $N$ in the case that $N \neq \{0\}$, we then assume, without loss of generality, that $|f| = 1$ a.e. on $T$. Then

$$N = \text{Ker}(I - H_f^* H_f)^{1/2} = \text{Ker}(T_f^* T_f)^{1/2} = \text{Ker}(T_f^* T_f) = \text{Ker} T_f.$$ 

By Lemma 5 we may write $f = h/h$ for some outer function $h$ in $H^2$, so $N$ is of the form in Theorem 6.
5. A parametrization of $S_u$. Let $S_u$ be defined as in §1. If $S_u$ is nonempty, then $S_u = S_{F/|F|}$ for some function $F$ in $H^1$. As before, we may assume, without loss of generality, that $F$ is outer. Let $h = F^{1/2}$. Now form the spaces $M_k$ as in §2 along with $b_k$ and $g_k$.

**Theorem 7.** Let $S_u$ be nonempty. If $h$, $b_k$, and $g_k$ are as above, then every $G$ in $S_u$ is of the form $G = (h/\bar{h})|g|^2$, where $||g||_2 = 1$ and $g$ is in the $L^2$ closure of $L^2 \cap g_0(H^2 \cap \mathbb{D})$.

**Proof.** We have $F/|F| = h/\bar{h} = b_0 g_0/\bar{g}_0$ in the notation of §2. If $G \in S_{F/|F|}$, then write $G = Bg^2$, where $g$ is outer in $H^2$ and $B$ is inner. Then $G/|G| = Bg/\bar{g} = h/\bar{h}$, so $Bg$ is seen to be in $H^2 \cap (h/\bar{h})H^2 = M_0$. Thus,

$$G = (h/\bar{h})|G| = (h/\bar{h})|Bg|^2.$$

Conversely, if $Bg \in M_0$, where $B$ is inner and $g$ is outer with unit norm in $H^2$, then $Bg = (h/\bar{h})B_1g$, where $B_1$ is inner, so $(h/\bar{h})|Bg|^2 = BB_1g^2$ and

$$\text{Arg}(BB_1g^2) = \text{Arg}(h/\bar{h}) = \text{Arg}(F/|F|),$$

so $(h/\bar{h})|Bg|^2$ belongs to $S_{F/|F|}$. The theorem now follows from Corollary 4.

It was noted in [2] that if $S_{F/|F|}$ contains a strong outer function and is not a singleton, then it contains functions with arbitrarily many zeroes in the unit disk $D$. This is a direct consequence of Theorem 1. It was also conjectured that, in this case, $S_{F/|F|}$ must contain a function whose inner part is not a finite Blaschke product. Corollary 4 provides an affirmative answer to this conjecture. To see this, note that if there is no bound for the number of zeroes for functions in $S_{F/|F|}$, then the related space $M_0$ must be infinite dimensional. Hence $b_0$ is not a finite Blaschke product, yet $b_0 g_0^2$ is in $S_{F/|F|}$.

6. Exposed points in $H^1$. It was noted by deLeuuw and Rudin in [3] that every exposed point $F$ in $H^1$ is a strong outer function. It was conjectured by Nakazi [6] that the converse also holds. Theorem 1 provides some evidence of this. Other sufficient conditions for $F$ to be exposed are discussed in [2]. An easy way to construct an outer function in $H^1$ which is not exposed is to take $F = (1 + B)^2$, where $B$ is an inner function. $F$ is outer since $1 + B$ takes its values in the right half-plane, and $F/|F| = B$, so $F$ is not exposed. Consider a Blaschke product $B$ with zero sequence $\{w_n\}$. It was shown by Ahern and Clark in [1] that $1 + B = (z - a)g$ for some $g$ in $H^2$ and $a \in T$ only if

$$\sum_{n=1}^{\infty} \frac{1 - |w_n|^2}{|1 - \bar{a}w_n|^2} < \infty.\tag{6.1}$$

However, there is an example (cited in [1]) due to Frostman [4] of a Blaschke product for which the sum in (6.1) diverges for every $a \in T$. It then follows that for this choice of $B$, $(1 + B)^2$ is a strong outer function which is not exposed. Hence the conjecture is false.

It would be of interest to have a usable characterization of the exposed points of $H^1$. The next theorem gives a characterization, though its usability is questionable.
THEOREM 8. Let \( h \) be an outer function in the unit sphere of \( H^2 \). Then \( h^2 \) is not exposed if and only if there exists a positive constant \( C < 1 \) such that

\[
|h(0)| \leq C\|T_{h/\overline{h}}(1 +zf)\|_2
\]

for every \( f \in H^2 \).

PROOF. Using the notation of §2, note that \( h \in M'_0 \) and \( h^2 \) is exposed if and only if \( M'_1 = \{0\} \). Now, \( M'_1 = \text{Ker} T_{h/\overline{h}} = (\text{Range} T_{h/\overline{h}})^\perp \). (See [7] for a discussion of Toeplitz operators.) Thus, if \( M'_1 = \{0\} \), \( h \in (M'_1)^\perp \), so \( h \) is in the closure of the range of \( T_{h/\overline{h}} \). In this case,

\[
1 = \|h\|_2 = \sup\{\langle h, T_{h/\overline{h}}g \rangle / \|T_{h/\overline{h}}g\|_2 \} = \sup\{\langle h, (h/\overline{h})g \rangle / \|T_{h/\overline{h}}g\|_2 \} = \sup\{|h(0)| : |g(0)| / \|T_{h/\overline{h}}g\|_2 \},
\]

where the supremum is taken over all \( g \in H^2 \) such that \( T_{h/\overline{h}}g \neq 0 \). Note that

\[
\langle h, T_{h/\overline{h}}g \rangle = \overline{h(0)} \cdot g(0),
\]

so if \( g(0) \neq 0 \), then \( T_{h/\overline{h}}g \neq 0 \). Hence, (6.2) can hold only if \( M'_1 \neq \{0\} \).

Suppose, conversely, that (6.2) fails to hold. Then \( h \) is orthogonal to \( M'_1 \) so, by Theorem 2, \( h \) is a scalar multiple of \( b_0g_0 \). But since \( h \) is outer, we must have that \( b_0 \) is a unimodular constant, and hence, by the same theorem, \( h^2 \) must be an exposed point of \( H^1 \). This establishes the theorem.

Finally, it seems that the key to characterizing the exposed points of \( H^1 \) lies in understanding the behavior of bounded outer functions in the left invariant subspaces \( H^2 \ominus bH^2 \) for arbitrary inner functions \( b \). For, suppose that \( F \) is outer in \( S \) and is not exposed. Take any other outer function \( G \) in \( S_F \ominus F \). We then have \((F + G)/2 = BK \in S_F \ominus F \), where \( B \) is a nonconstant inner function and \( K \) is outer (this is because \((F + G)/2 \) is not an extreme point of \( S \); see [3]). Note that \(|K| \geq |G|/2 \) and \(|K| \geq |F|/2 \) a.e. on \( T \) since \( F \) and \( G \) have the same argument a.e. Thus, \( F = 2K(B - G/2K) \). Now \( \text{Arg}(G/2K) = \text{Arg}(B) \) a.e., and \(|G/2K| < 1 \) a.e., and \( B - G/2K \) is outer since \( F \) is. Note further that \(|B - G/2K| = 1 - |G/2K| \). Thus we may write \( F = 2Kg^2 \), where \( g \) is outer in \( H^2 \ominus zBH^2 \) and \( 0 < |g| < 1 \) a.e. on \( T \). Of course, when \( B \) turns out to be a finite Blaschke product, \( g^2 \) has double zeroes on \( T \) by Theorem 1.

ADDED IN PROOF. The author can now show that \( M_0 = g_0(H^2 \ominus zb_0H^2) \).

REFERENCES

6. ____*, The kernels of Toeplitz operators, preprint.

DEPARTMENT OF MATHEMATICS, FORDHAM UNIVERSITY, BRONX, NEW YORK 10458

Current address:    Department of Mathematics, San Francisco State University, 1600 Holloway Avenue, San Francisco, California 94132

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use