ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF VOLterra INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. The asymptotic behavior of solutions of Volterra integro-differential equations of the form

\[ x'(t) = A(t)x(t) + \int_0^t K(t,s)x(s)\,ds + F(t) \]

is discussed in which \( A \) is not necessarily a stable matrix. An equivalent equation which involves an arbitrary function is derived and a proper choice of this function would pave a way for the new coefficient matrix \( B \) (corresponding to \( A \)) to be stable.

1. Introduction. The objective of this paper is to investigate the asymptotic behavior of solutions of the Volterra integro-differential equation (VIDE)

\[ x'(t) = A(t)x(t) + \int_0^t K(t,s)x(s)\,ds + F(t) \]  

(1.1)

where \( A(t) \) and \( K(t,s) \) are \( n \times n \) matrices defined and continuous on \( 0 \leq t < \infty \) and \( 0 \leq s \leq t < \infty \), respectively, and \( x(t) \) and \( F(t) \) are \( n \)-vectors with \( F(t) \) continuous on \( 0 \leq t < \infty \), when the matrix \( A \) is not necessarily stable. Our main approach here is by way of deriving an equivalence theorem (Lemma 2.1) which has the potential to supply us a stable matrix \( B \) corresponding to \( A \).

It is well known that the linear autonomous ordinary differential system is asymptotically stable if all the characteristic roots of the coefficient matrix have negative real parts [12, Chapter 3]. For nonautonomous systems, with an addition of the Lipschitz condition on the coefficient matrix, similar results have been expounded in [4 and 5]. Thus while studying VIDE (1.1), be it through Liapunov second method [1, 2, 14, 15] or from perturbation theory [8, 10, 13], it has invariably been assumed that the coefficient matrix is stable. Notable exceptions that have dispensed with the stability condition on the coefficient matrix have been the works of Levin [9], Grossman and Miller [7], Grimmer and Seifert [6], Burton [3], among others. In [9] this has been done by defining a suitable energy function while in [7] the integrability of the resolvent function of VIDE (1.1) has been characterized by a transformation condition similar to that given in [11] for Volterra integral equations. In [6] the same has been achieved by studying the resolvent of a transformed equation. Quite recently in [3], the conditions involving the anti-derivatives of the kernel are assumed. Motivated by the interesting nature of this problem, an attempt has been made in §2 to study the asymptotic behavior of solutions of (1.1)
when the coefficient matrix $A$ in (1.1) is not necessarily stable. Our approach here has been to develop an equivalent equation which involves an arbitrary function. A proper choice of this function would pave a way for the new coefficient matrix (corresponding to equivalent VIDE) to be stable.

2. Main results. The following lemmas are useful in our subsequent discussion.

**Lemma 2.1.** Let $(t, s)$ be an $n \times n$ continuously differentiable matrix function on $0 \leq s \leq t < \infty$. Then the equation (1.1) is equivalent to

\[
y'(t) = B(t)y(t) + \int_0^t L(t, s)y(s) \, ds + H(t), \quad y(0) = x_0,
\]

where

\[
B(t) = A(t) - \Phi(t, t),
\]

\[
L(t, s) = K(t, s) + \int_s^t \Phi(t, u)K(u, s) \, du,
\]

and

\[
H(t) = F(t) + \int_0^t \Phi(t, 0)x_0 + \int_0^t \Phi(t, s)F(s) \, ds.
\]

**Proof.** Let $x(t)$ with $x(0) = x_0$ be any solution of (1.1) existing on the interval $0 \leq t < \infty$. Consider the identity

\[
\int_0^t \Phi(t, s)x(s) \, ds = \Phi(t, t)x(t) - \Phi(t, 0)x_0 - \int_0^t \Phi(t, s)x'(s) \, ds.
\]

Substituting for $x'(t)$ from (1.1) and using Fubini's theorem, we get

\[
\int_0^t \Phi(t, s)x(s) \, ds = \Phi(t, t)x(t) - \Phi(t, 0)x_0 - \int_0^t \Phi(t, s)x'(s) \, ds
\]

\[
\quad - \int_0^t \Phi(t, s)A(s)x(s) \, ds - \int_0^t \int_0^t \Phi(t, s)K(s, \tau) \, ds \, x(\tau) \, d\tau.
\]

Then it follows from (1.1) and (2.1)–(2.3) that

\[
\int_0^t L(t, s)x(s) \, ds = \int_0^t K(t, s)x(s) \, ds + \int_0^t \Phi(t, s)x(s) \, ds
\]

\[
\quad + \int_0^t \Phi(t, s)A(s)x(s) \, ds
\]

\[
\quad + \int_0^t \int_0^t \Phi(t, u)K(u, s) \, du \, x(s) \, ds
\]

\[
\quad = x'(t) - A(t)x(t) - F(t) + \Phi(t, t)x(t)
\]

\[
\quad - \Phi(t, 0)x_0 - \int_0^t \Phi(t, s)F(s) \, ds
\]

\[
\quad = x'(t) - B(t)x(t) + H(t).
\]

Thus every solution of (1.1) is also a solution of (2.1). Conversely, let $y(t)$ be a solution of (2.1) with $y(0) = x_0$. Define

\[
z(t) = y'(t) - F(t) - A(t)y(t) - \int_0^t K(t, s)y(s) \, ds.
\]
From (2.1), (2.2) and the definition of \( z(t) \) we obtain
\[
z(t) = -\Phi(t, t)y(t) + \int_0^t \left[ \Phi_s(t, s) + \Phi(t, s)A(s) + \int_s^t \Phi(t, u)K(u, s) \, du \right] y(s) \, ds \\
+ \Phi(t, 0)x_0 + \int_0^t \Phi(t, s)F(s) \, ds.
\]
Substituting for \( F(s) \) from the definition of \( z(t) \) and changing the order of integration, we get
\[
z(t) = -\Phi(t, t)y(t) - \Phi(t, 0)x_0 - \int_0^t \Phi(t, s)y'(s) \, ds \\
+ \int_0^t \Phi_s(t, s)y(s) \, ds - \int_0^t \Phi(t, s)z(s) \, ds.
\]
From the identity
\[
\int_0^t \Phi(t, s)y'(s) \, ds = \Phi(t, t)y(t) - \Phi(t, 0)x_0 - \int_0^t \Phi_s(t, s)y(s) \, ds
\]
it is clear that
\[
z(t) = -\int_0^t \Phi(t, s)z(s) \, ds.
\]
Since \( \Phi(t, s) \) is continuous, it follows from the uniqueness of solutions of Volterra integral equations that \( z(t) = 0 \). Hence \( y(t) \) solves (1.1).

**Remark 2.2.** It is to be noted that if \( \Phi(t, s) \) is the differentiable resolvent corresponding to the kernel \( K(t, s) \), then the equation (2.1) together with (2.2) gives the usual variation of constants formula (see Grossman and Miller [8]).

**Lemma 2.3.** Let \( B(t) \) be an \( n \times n \) continuous matrix which commutes with its integral and let \( M \) and \( \alpha \) be positive numbers. Suppose the inequality
\[
(2.4) \quad \left| \exp \left( \int_s^t B(\tau) \, d\tau \right) \right| \leq Me^{-\alpha(t-s)}, \quad 0 \leq s \leq t < \infty,
\]
holds. Then every solution \( x(t) \) of (2.1) with \( x(0) = x_0 \) satisfies
\[
(2.5) \quad |x(t)| \leq M|x_0|e^{-\alpha t} + M \int_0^t e^{-\alpha(t-\tau)}|H(\tau)| \, d\tau \\
+ M \int_0^t \left[ \int_0^t e^{-\alpha(t-\tau)}|L(\tau, s)| \, d\tau \right] |x(s)| \, ds.
\]

**Proof.** Multiplying both sides of (2.1) by \( \exp(-\int_0^t B(\tau) \, d\tau) \) and rearranging the terms, we obtain
\[
\left( \exp \left( -\int_0^t B(\tau) \, d\tau \right) x(t) \right)' = \exp \left( -\int_0^t B(\tau) \, d\tau \right) \left[ H(t) + \int_0^t L(t, s)x(s) \, ds \right].
\]
Integrating from 0 to \( t \), we get
\[
\exp \left( -\int_0^t B(\tau) \, d\tau \right) x(t) = x_0 + \int_0^t \exp \left( -\int_0^s B(\tau) \, d\tau \right) H(s) \, ds \\
+ \int_0^t \exp \left( -\int_0^s B(\tau) \, d\tau \right) \left( \int_0^s L(s, u)x(u) \, du \right) \, ds.
\]
By changing the order of integration on the right side and using (2.4), we obtain (2.5).

**Remark 2.4.** If \( B \) is a constant matrix, then it commutes with its integral. Further, the condition (2.4) holds if, in addition, all the characteristic roots of \( B \) have negative real parts (cf. [12, Chapter 2]).

**Theorem 2.5.** Let \( \Phi(t, s) \) be a continuously differentiable \( n \times n \) matrix function such that, for \( 0 \leq s \leq t < \infty \),
- (i) the hypotheses of Lemma 2.3 holds,
- (ii) \( |\Phi(t, s)| \leq L_0 e^{-\gamma (t-s)} \),
- (iii) \( \sup_{0 \leq s \leq t < \infty} \int_s^t e^{\alpha (t-s)} |L(\tau, s)| d\tau \leq \alpha_0 \),

where \( L_0, \gamma (> \alpha), \alpha_0 \) are positive real numbers. Suppose further
- (iv) \( F(t) \equiv 0 \),
where \( F(t) \) is defined in (1.1). If \( \alpha - M_\alpha > 0 \), then every solution \( x(t) \) of (1.1) tends to zero exponentially as \( t \to +\infty \).

**Proof.** In view of Lemma 2.1 and the function \( \Phi(t, s) \) satisfying the conditions (i), (ii) and (iii), it is enough to show that every solution of (2.1) tends to zero exponentially as \( t \to +\infty \). Since \( F(t) \equiv 0 \), the equation (2.2) and (2.5) and the condition (ii) imply that
\[
e^{\alpha t} |x(t)| \leq M|x_0| + M L_0 |x_0| \int_0^t e^{-(\gamma - \alpha) \tau} d\tau + M \int_0^t \left[ \int_s^t e^{\alpha \tau} |L(\tau, s)| d\tau \right] |x(s)| ds.
\]
Using (iii), we get
\[
e^{\alpha t} |x(t)| \leq M|x_0| + \frac{M L_0 |x_0|}{(\gamma - \alpha)} + \int_0^t M \alpha_0 e^{\alpha s} |x(s)| ds.
\]
The application of Gronwall inequality yields that
\[
e^{\alpha t} |x(t)| \leq M|x_0| \left( 1 + \frac{L_0}{(\gamma - \alpha)} \right) e^{M_\alpha_0 t}.
\]
This implies that
\[
|x(t)| \leq M|x_0| \left( 1 + \frac{L_0}{(\gamma - \alpha)} \right) e^{-(\alpha - M_\alpha_0) t}.
\]
Thus in view of \( \alpha - M_\alpha_0 > 0 \), the result follows.

**Corollary 2.6.** In addition to the assumptions (i), (ii) and (iv) of Theorem 2.5, suppose the following conditions hold:
- (a) \( |K(t, s)| \leq K_0 e^{-\beta (t-s)} \) for \( 0 \leq s \leq t < \infty \),
- (b) \( |\Phi(t, s)| \leq N_0 e^{-\delta (t-s)} \) for \( 0 \leq s \leq t < \infty \),
- (c) \( |A(t)| \leq A_0 \) for \( 0 \leq t < \infty \)

where \( A_0, N_0, K_0, \beta, \delta \) are positive real numbers, and
- (d) \( \gamma > \beta > \alpha, \delta > \alpha \) and \( \alpha - M_\alpha_0 > 0 \)
where
\[
\alpha_0 \overset{\text{def}}{=} \left[ \frac{K_0}{\beta - \alpha} + \frac{N_0}{\delta - \alpha} + \frac{L_0 A_0}{\gamma - \alpha} + \frac{K_0 L_0}{(\beta - \alpha)(\gamma - \beta)} \right].
\]

Then every solution \( x(t) \) of (1.1) tends to zero exponentially as \( t \to +\infty \).
PROOF. Following the proof of Theorem 2.5, we obtain

\begin{equation}
(2.6) \quad e^{\alpha t}|x(t)| \leq M|x_0| \left(1 + \frac{L_0}{(\gamma - \alpha)}\right) + M \int_0^t \left[ \int_s^t e^{\alpha \tau} \left| K(\tau, s) + \Phi_s(\tau, s) + \Phi(\tau, s)A(s) \right| + \int_s^t \Phi(\tau, u)K(u, s) \, du \right] |x(s)| \, ds.
\end{equation}

Using conditions (i), (a), (b), (c) and estimating each integral on the right side of (2.6), we get

\begin{equation}
(2.7) \quad e^{\alpha t}|x(t)| < M|x_0|(1 + \frac{L_0}{(\gamma - \alpha)}) + \int_0^t \tilde{M}\alpha_0 e^{\alpha s} |x(s)| \, ds.
\end{equation}

Thus, in view of condition (d), the application of Gronwall’s inequality yields the desired result.

**Remark 2.7.** If \( F(t) = 0 \) in equation (1.1), then the Theorem 2.5 asserts that the zero solution of (1.1) is exponentially asymptotically stable.

**Remark 2.8.** If \( F(t) \) is not zero in Theorem 2.5, still the solutions of (1.1) tends to zero as \( t \to +\infty \) provided \( \int_0^\infty |F(s)| \, ds < \infty \). This is an immediate consequences of variation of constants formula (see [8]) and Theorem 2.5.

**Remark 2.9.** It is possible to select a matrix function \( \Phi(t, s) \) satisfying the conditions (i) and (ii) of Theorem 2.5 and condition (b) of Corollary 2.6. For example, if \( \Phi(t, s) = L_0 e^{-\gamma(t-s)}I \), then \( N_0 = L_0 \gamma \) and \( \delta = \gamma \). \( \Phi(t, t) \) being a constant matrix in this case, the estimate (2.4) is guaranteed if \( A(t) \) is a constant matrix and \( B \) is negative definite.

**Remark 2.10.** Basically it is the condition “\( \tilde{M}\alpha_0 < \alpha \)” in Corollary 2.6 which controls the asymptotic nature of the solution \( x(t) \) of (1.1). A look at the composition of \( \tilde{\alpha}_0 \) reveals that while so choosing \( \gamma \) and \( \delta \) much away from \( \beta \) and \( \alpha \), respectively, we can nullify the effect of the last three terms in \( \tilde{\alpha}_0 \), the first term \( K_0/(\beta - \alpha) \) being the essential term which we have to reckon with. Therefore, if \( \beta \) is so large as to exceed \( (\alpha^2 + MK_0)/\alpha \), then \( \tilde{M}\alpha_0 \) would be less than \( \alpha \). Thus we see that the attenuation required on the kernel \( K(t, s) \) is linked with the constant \( \alpha \) in (2.4). This conclusion implicidy assumed the estimate (2.4). Such an estimate would be possible when the transformed matrix \( B \) is constant and negative definite.

**Remark 2.11.** In [3], a condition of the type (2.4) has been used for the matrix \( Q \equiv (A(t) - G(t, t)) \), where \( G(t, s) \) is the anti-derivative of the kernel \( K(t, s) \) (i.e. \( \partial G(t, s)/\partial t = K(t, s) \)). As such the matrix \( B \) in our study allows more flexibility due to the arbitrary character of the function \( \Phi(t, s) \). Further, our approach is entirely different and the analysis in [3] can be applied to equation (2.1) in order to obtain sharper estimates. Thus our Theorem 2.5 is in addition to the Theorem 2 of [3] rather than a substitute for it.

**Example 2.12.** In (1.1) (scalar case), let \( A(t) = a_1 e^{-b_1 t} - a_2 \), \( K(t, s) = e^{-b_2 (t, s)} \) and \( F(t) \equiv 0 \) where \( a_1, a_2, b_1, b_2 \) are positive real numbers. Choose
\( \Phi(t,s) = a_1 e^{-b_1 t} \). Then \( M = 1, \alpha = a_2, A_0 = a_1 + a_2, L_0 = a_1, K_0 = 1, \beta = b_2, \gamma = \delta = b_1 \) and \( N_0 = 0 \). Thus the condition (d) of Corollary 2.6 holds if

\[
\begin{align*}
& a_1 = \dot{K} a_2, \\
& b_1 = (4\dot{K}^2 + 4\dot{K} + 1)a_2, \\
& b_2 = (4 + a_2^2)/a_2 \quad \text{and} \quad a_2 \geq 2/[\dot{K}(4\dot{K} + 3)]^{1/2}
\end{align*}
\]

where \( \dot{K} (1 < \dot{K} < \infty) \) is an arbitrary real number. For example, if \( \dot{K} = 2, a_2 = 1, \) then \( M_{\alpha_0} \approx 0.53 \) and \( \alpha = 1 \).

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