

AN OSCILLATION CRITERION FOR LINEAR SECOND-ORDER DIFFERENTIAL SYSTEMS

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ABSTRACT. This article is concerned with the oscillatory behavior at infinity of the solution $y: [a, \infty) \rightarrow \mathbf{R}^n$ of a system of n second-order differential equations, $y''(t) + Q(t)y(t) = 0$, $t \in [a, \infty)$; Q is a continuous matrix-valued function on $[a, \infty)$ whose values are real symmetric matrices of order n .

It is shown that the solution is oscillatory at infinity if (at least) $n - 1$ eigenvalues of the matrix $\int_a^t Q(t) dt$ tend to infinity as $t \rightarrow \infty$.

1. Introduction. We are concerned with the differential equation

$$(1.1) \quad y''(t) + Q(t)y(t) = 0, \quad t \in [a, \infty),$$

for a vector-valued function $y: [a, \infty) \rightarrow \mathbf{R}^n$. Here Q is a continuous matrix-valued function on $[a, \infty)$ whose values are real symmetric matrices of order n .

Two points $\alpha, \beta \in [a, \infty)$ are said to be conjugate relative to (1.1) if there exists a nontrivial function y which satisfies (1.1) and vanishes at α and β . (1.1) is said to be oscillatory at infinity if, for any point $\alpha \in [a, \infty)$, there exists a point $\beta \in (\alpha, \infty)$ such that α and β are conjugate relative to (1.1). We use the notation $Q_1(t)$ for the matrix of the integrals over $[a, t]$ of the corresponding elements of $Q(t)$,

$$(1.2) \quad Q_1(t) = \int_a^t Q(s) ds.$$

The oscillation theory for (1.1) has received considerable attention in the past; see, for example, Reid [7, Chapter V]. It has been conjectured that (see Hinton and Lewis [1]) (1.1) is oscillatory at infinity whenever

$$(1.3) \quad \lim_{t \rightarrow \infty} \lambda_1\{Q_1(t)\} = \infty.$$

Here, $\lambda_1\{\cdot\}$ denotes the largest eigenvalue of the matrix inside the braces. At this point, the conjecture has been established for the case $n = 2$ by Kwong and Kaper [2]. For arbitrary n , the conjecture has been established under various growth conditions on the trace of $Q_1(t)$ by Mingarelli [5, 6] and Kwong et al. [3].

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We shall study the oscillatory behavior of (1.1) by means of the Riccati equation for the matrix-valued function R ,

$$(1.4) \quad R(t) = R(a) + Q_1(t) + \int_a^t R^2(s) ds, \quad t \in [a, \infty).$$

A necessary and sufficient condition for (1.1) to be nonoscillatory at infinity is that (1.4) has a continuous solution R on a half-line (which, without loss of generality, may be taken as $[a, \infty)$) whose values are selfadjoint matrices of order n ; see [2, 3].

In this paper we prove that (1.1) is oscillatory at infinity if (at least) $n - 1$ eigenvalues of $Q_1(t)$ tend to infinity as $t \rightarrow \infty$,

$$(1.5) \quad \lim_{t \rightarrow \infty} \lambda_i \{Q_1(t)\} = \infty, \quad i = 1, \dots, n - 1.$$

Here each eigenvalue is counted according to its multiplicity. The proof is given in §3; §2 contains some preliminary material.

2. Preliminaries. Our main goal in this section is to establish an ordering relation for the quadratic term in the matrix Riccati equation

$$(2.1) \quad R(t) = F(t) + \int_0^t R^2(s) ds, \quad t \geq 0.$$

Here F is a given function defined on $(0, \infty)$, whose values are symmetric matrices of order n . We shall use the standard partial ordering in the space of selfadjoint matrices, viz., $A \geq B$ if $A - B$ is nonnegative. The symbol I stands for the identity matrix of order n .

At several points in the discussion we shall need the following result from matrix theory.

LEMMA 1. *If A is a real symmetric matrix of order n with eigenvalues $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n$, B is a nonnegative matrix of order n , and $A + B$ has eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$, then $\lambda_i \geq \mu_i$ for $i = 1, \dots, n$.*

PROOF. The lemma is a consequence of the Courant-Fischer theorem; see [4, Theorem 3.6.3]. \square

The next lemma contains an auxiliary result that we shall need in the proof of Lemma 3.

LEMMA 2. *Let $A = (a_{ij})$ be a real symmetric matrix of order $n \geq 2$ such that $A \geq 0$. If for some $k \in [2^{-1/2}, 1)$ the matrix $B = (b_{ij})$, with elements*

$$\begin{aligned} b_{ij} &= k(a_{ij} + 1) & \text{if } i = j, \\ b_{ij} &= a_{ij} & \text{if } i \neq j, \end{aligned}$$

has $n - 1$ eigenvalues ≤ 0 (each eigenvalue counted according to its multiplicity), then $a_{ii} > 1$ for $i = 1, \dots, n$.

PROOF. Consider first the case $n = 2$. Because $\text{tr } B = k(2 + \text{tr } A)$ and $A \geq 0$, we have $\text{tr } B > 0$, so at least one eigenvalue of B must be positive. Hence, since the other eigenvalue is negative or zero, $\det B \leq 0$, i.e., $k^2(a_{11} + 1)(a_{22} + 1) \leq a_{12}^2$. But

$A \geq 0$, so $a_{12}^2 \leq a_{11}a_{22}$. Therefore, $a_{11} > 0$ and $a_{22} > 0$, and $k^2(1 + a_{11}^{-1})(1 + a_{22}^{-1}) \leq 1$. This inequality implies

$$k^2(1 + a_{11}^{-1}) < 1 \quad \text{and} \quad k^2(1 + a_{22}^{-1}) < 1,$$

whence $a_{11} > 1$ and $a_{22} > 1$.

In the general case we have $\text{tr } B = k(n + \text{tr } A) > 0$, so the largest eigenvalue of B must be positive, while all other eigenvalues are negative or zero. Hence, the quadratic form, Q say, associated with the matrix B is negative or at most equal to zero on a subspace Σ of \mathbf{R}^n of dimension $n - 1$. Now, consider any of the matrices

$$B_{ij} = \begin{pmatrix} b_{ii} & b_{ij} \\ b_{ij} & b_{jj} \end{pmatrix}, \quad i, j = 1, \dots, n.$$

If both eigenvalues of B_{ij} were positive, the quadratic form Q would be positive on a two-dimensional subspace of \mathbf{R}^n . This subspace would certainly have a nontrivial intersection with the subspace Σ introduced above, where Q is negative or at most equal to zero. This contradiction leads us therefore to conclude that $\det B_{ij} \leq 0$, i.e., $k^2(1 + a_{ii})(1 + a_{jj}) \leq a_{ij}^2$. The proof is now completed as before. \square

The following lemma gives the ordering result for the quadratic term in the Riccati equation (2.1).

LEMMA 3. *Let F be a given function defined on $(0, \infty)$, whose values are selfadjoint matrices of order n with entries in $L^1_{\text{loc}}([0, \infty))$. Let T be chosen such that $T > 16n^2$. Suppose the matrix-valued function R satisfies the Riccati equation (2.1) on $[0, T]$. If, for every $t \in [0, T]$, at least $n - 1$ eigenvalues of $F(t)$ are ≥ 1 , then*

$$(2.2) \quad \int_0^T R^2(s) \, ds \geq I.$$

PROOF. The matrix $R(t)$ is symmetric for all $t \in [0, T]$. Without loss of generality we may assume that the matrix $\int_0^T R^2(s) \, ds$ is diagonal. We use the following notation:

$$R(t) = (r_{ij}(t)), \quad P(t) = \int_0^t R^2(s) \, ds = (\rho_{ij}(t)).$$

We cover the interval $[0, T]$ by sets $S_i, i = 0, \dots, n$, as follows:

$$S_i = \left\{ t \in [0, T] : \sum_{j=1}^n |r_{ij}(t)| \geq \frac{1}{4}(1 + \rho_{ii}(t)) \right\}, \quad i = 1, \dots, n,$$

$$S_0 = [0, T] \setminus \bigcup_{i=1}^n S_i.$$

On $S_i (i = 1, \dots, n)$ we have

$$\rho'_{ii} = \sum_{j=1}^n |r_{ij}|^2 \geq \frac{1}{n} \left(\sum_{j=1}^n |r_{ij}| \right)^2 \geq \frac{1}{16n} (1 + \rho_{ii})^2,$$

so ρ_{ii} is (strictly) increasing, $(1 + \rho_{ii})^{-1}$ is decreasing, and $-(d/dt)(1 + \rho_{ii})^{-1} \geq 1/16n$. Upon integration over S_i we obtain the estimate

$$\text{mes } S_i \leq 16n, \quad i = 1, \dots, n.$$

Because $T > 16n^2$, we conclude that S_0 is nonempty.

For any t we have $R(t) \leq \tilde{R}(t)$, where $\tilde{R}(t)$ is the diagonal matrix with entries $\sum_{j=1}^n |r_{ij}(t)|$. (Note that $\tilde{R}(t) - R(t)$ is a symmetric diagonally dominant matrix with nonnegative diagonal entries; cf. [4, §7.2].) In particular, if $t \in S_0$, then $\sum_{j=1}^n |r_{ij}| < \frac{1}{4}(1 + \rho_{ii})$ for each i , so $R(t) \leq \frac{1}{4}(I + \tilde{P}(t))$, where $\tilde{P}(t)$ is the diagonal matrix with entries $\rho_{ii}(t)$. Thus, $I - F(t) = I + P(t) - R(t) \geq I + P(t) - \frac{1}{4}(I + \tilde{P}(t))$. Since at least $n - 1$ eigenvalues of $I - F(t)$ are ≤ 0 , the same is true for the matrix $I + P(t) - \frac{1}{4}(I + \tilde{P}(t))$. (Here we have used Lemma 1.) On the other hand,

$$\text{tr}(I + P(t) - \frac{3}{4}(I + \tilde{P}(t))) = \frac{3}{4}(n + \text{tr } P(t)) > 0,$$

so the matrix $I + P(t) - \frac{1}{4}(I + \tilde{P}(t))$ has at least one positive eigenvalue. We conclude that this matrix has precisely one positive eigenvalue and $n - 1$ eigenvalues which are negative or zero.

Applying Lemma 2 to the matrices $A = P(t)$ and $B = I + P(t) - \frac{1}{4}(I + \tilde{P}(t))$ with $k = \frac{3}{4}$, we see that $\rho_{ii}(t) > 1$ for $i = 1, \dots, n$, whenever $t \in S_0$. As each ρ_{ii} is a nondecreasing function of t , the same inequalities must hold at $t = T$. Because $P(T)$ is diagonal, (2.2) follows. \square

3. Main result. The ordering relation given in Lemma 3 enables us to compare the quadratic term in (1.4) with multiples of the identity matrix and thus to show that (1.4) does not have a solution on $[a, \infty)$ if (1.5) holds.

THEOREM. *If (1.5) holds, then (1.1) is oscillatory at infinity.*

PROOF. The proof is by contradiction, where we assume that the Riccati equation (1.4) has a selfadjoint matrix-valued solution R on $[a, \infty)$.

If (1.5) holds, then $\lambda_i\{R(a) + Q_1(t)\} \rightarrow \infty$ as $t \rightarrow \infty$ for $i = 1, \dots, n - 1$, so for t sufficiently large, $t \geq t_0$ say, we have

$$(3.1) \quad \lambda_i\{R(a) + Q_1(t)\} \geq 1, \quad i = 1, \dots, n - 1, t \geq t_0.$$

Let t_0 be kept fixed. For $t \geq t_0$ we have

$$(3.2) \quad R(t) = F_0(t) + \int_{t_0}^t R^2(s) ds, \quad t \geq t_0,$$

where

$$(3.3) \quad F_0(t) = R(a) + Q_1(t) + \int_a^{t_0} R^2(s) ds, \quad t \geq t_0.$$

The inequalities (3.1) and the nonnegativity of the integral term imply

$$(3.4) \quad \lambda_i\{F_0(t)\} \geq 1, \quad i = 1, \dots, n - 1, t \geq t_0.$$

The change of variables $\bar{R}(t) = R(t_0 + t)$ reduces (3.2) to

$$(3.5) \quad \bar{R}(t) = \bar{F}(t) + \int_0^t \bar{R}^2(s) ds, \quad t \geq 0,$$

where $\bar{F}(t) = F_0(t_0 + t)$. It follows from (3.4) that $n - 1$ eigenvalues of $\bar{F}(t)$ are ≥ 1 , so according to Lemma 2 there exists a $\tau > 0$ such that $\int_0^\tau \bar{R}^2(s) ds \geq I$. Let $t_1 = t_0 + \tau$ thus determined be kept fixed. Then

$$(3.6) \quad \int_{t_0}^t R^2(s) ds \geq I, \quad t \geq t_1.$$

Proceeding to the next step, we define

$$(3.7) \quad F_1(t) = F_0(t) + \int_{t_0}^{t_1} R^2(s) ds, \quad t \geq t_1.$$

Because of (3.4) and (3.6) we have

$$(3.8) \quad \lambda_i \{ F_1(t) \} \geq 2, \quad i = 1, \dots, n - 1; t \geq t_1.$$

Furthermore, from (3.2) it follows that

$$(3.9) \quad R(t) = F_1(t) + \int_{t_1}^t R^2(s) ds, \quad t \geq t_1.$$

The change of variables $\bar{R}(t) = \frac{1}{2}R(t_1 + \frac{1}{2}t)$ transforms this equation into (3.5), where $\bar{F}(t) = \frac{1}{2}F_1(t_1 + \frac{1}{2}t)$. Thus, $\lambda_i \{ \bar{F}(t) \} \geq 1$ for $i = 1, \dots, n - 1$ and Lemma 2 applies. Adjusting the value of τ if necessary, we conclude that $\int_0^{\bar{R}^2} R^2(s) ds \geq I$ for $t \geq \tau$, i.e.,

$$(3.10) \quad \int_{t_1}^t R^2(s) ds \geq I, \quad t \geq t_2,$$

where $t_2 = t_1 + \frac{1}{2}\tau = t_0 + \frac{3}{2}\tau$.

Continuing this process we find, after n steps,

$$(3.11) \quad \int_{t_{n-1}}^t R^2(s) ds \geq I, \quad t \geq t_n,$$

where $t_n = t_{n-1} + \tau/2^{n-1} = t_0 + 2(1 - 2^{-n})\tau$. Thus, adding the contributions of $[a, t_0], [t_0, t_1], \dots, [t_{n-1}, t_n]$, we obtain the estimate

$$(3.12) \quad \int_a^t R^2(s) ds \geq (2^n - 1)I, \quad t \geq t_n.$$

But t_n tends to the finite limit $t_0 + 2\tau$ as $n \rightarrow \infty$, so there exists a finite number T such that $R(t)$ blows up as $t \uparrow T$. This conclusion contradicts the assumption that (1.4) has a solution on $[a, \infty)$. \square

A closer examination of the proof of Lemma 3 shows that all that is needed for the lemma to hold is that enough (namely, $n - 1$) eigenvalues of the matrix $I + P(t) - \frac{1}{4}(I + \dot{P}(t))$ are less than or equal to zero. The condition that $n - 1$ eigenvalues of $F(t)$ are greater than or equal to one is sufficient, but by no means necessary for this to be true. Consequently, the condition (1.5) is only sufficient for (1.1) to be oscillatory at infinity.

Also, the condition (1.5) need not hold for all sufficiently large tn . If the eigenvalues grow beyond any bound on a sufficiently large set of t -values, the conclusion remains valid. We refer the reader to our earlier publication [2], where this idea is pursued in some detail.

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