SUMMING GENERALIZED CLOSED \(U\)-SETS FOR WALSH SERIES

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Abstract. A countable union of closed \(U\)-sets for Walsh series in certain generalized sense is again a \(U\)-set in the same sense.

1. Introduction. Let \( \mu \sim \sum_{k=0}^{\infty} \hat{\mu}(k)w_k(x) \) be a Walsh series. A subset \( E \) of the dyadic group is said to be a \( U \)-set if

\[
\sum_{k=0}^{\infty} \hat{\mu}(k)w_k(x) = 0 \quad \text{everywhere except on } E
\]

implies that \( \mu \) is the zero series.

Wade [2] proved that if \( E_1, E_2, \ldots \) are closed \( U \)-sets, then \( \bigcup_{k=1}^{\infty} E_k \) is also a \( U \)-set.

In this paper we shall generalize Wade's theorem. Let \( \mathcal{A} \) be a certain class of Walsh series. A subset \( E \) of the dyadic group is said to be a \( U \)-set for \( \mathcal{A} \) if \( \mu \in \mathcal{A} \) and

\[
\lim_{n \to \infty} \frac{1}{2^n} \sum_{k=0}^{2^n-1} \hat{\mu}(k)w_k(x) = 0 \quad \text{everywhere except on } E
\]

imply that \( \mu \) is the zero series. We have already proved in [3] that when \( E \) is a closed subset of the dyadic group, (1) holds if and only if (2) holds and

\[
\hat{\mu}(k) = o(1) \quad \text{as } k \to \infty.
\]

Therefore a closed subset of the dyadic group is a \( U \)-set in the classical sense if and only if it is a \( U \)-set for the class of Walsh series \( \mu \) which satisfies (3).

When \( \mathcal{A} \) satisfies the following conditions, we say that \( \mathcal{A} \) satisfies the condition (L):

(i) a \( U \)-set for \( \mathcal{A} \) is of measure zero;
(ii) if \( \mu \in \mathcal{A} \), then

\[
\liminf_{n \to \infty} \left| \frac{1}{2^n} \sum_{k=0}^{2^n-1} \hat{\mu}(k)w_k(x) \right| = 0 \quad \text{everywhere};
\]

(iii) if \( \mu \) and \( \mu' \in \mathcal{A} \), then \( \alpha \mu + \alpha' \mu' \in \mathcal{A} \) for arbitrary real numbers \( \alpha \) and \( \alpha' \), where

\[
(\alpha \mu + \alpha' \mu') \sim \sum_{k=0}^{\infty} (\alpha \mu + \alpha' \mu')\hat{(k)}w_k(x)
\]

\[
= \sum_{k=0}^{\infty} (\alpha \hat{\mu}(k) + \alpha' \hat{\mu}'(k))w_k(x);
\]

Received by the editors June 18, 1984.

1980 Mathematics Subject Classification. Primary 42C25.

Key words and phrases. Uniqueness, Walsh series.
(iv) if \( \mu \in \mathcal{A} \), then
\[
\sum_{k=0}^{\infty} \hat{\mu}(k+j)w_k(x) \in \mathcal{A} \quad \text{for} \quad j = 1, 2, \ldots.
\]

We shall prove the following theorem.

**Theorem 1.** When a class of Walsh series \( \mathcal{A} \) satisfies the condition (L) and if \( E_1, E_2, \ldots \) are closed \( U \)-sets for \( \mathcal{A} \), then \( \bigcup_{k=1}^{\infty} E_k \) is also a \( U \)-set for \( \mathcal{A} \).

2. Notations and lemmas. In this paper we shall use the following notations. Let \( I_n^p \) be the set of all 0-1 sequences, \( (t_1, t_2, \ldots) \), such that \( \sum_{k=1}^{n} t_k/2^k = p/2^n \). \( I_n^p \) is called a dyadic interval of rank \( n \). For convenience, \( I_n(x) \) denotes the dyadic interval of rank \( n \) containing \( x \). A dyadic interval is closed and open. We refer the details of the dyadic group, Walsh functions, the operation \( \pm \) and so on to Fine’s paper [1].

**Lemma 2.** When \( \mathcal{A} \) satisfies the condition (L), if \( \mu \in \mathcal{A} \) and \( I \) is a dyadic interval, then there exists a Walsh series \( \mu^* \in \mathcal{A} \) which satisfies the following conditions:

(i) \[
\lim_{n \to \infty} \left| \sum_{k=0}^{2^n-1} \hat{\mu}(k)w_k(x) - \sum_{k=0}^{2^n-1} \hat{\mu}^*(k)w_k(x) \right| = 0 \quad \text{on} \quad I,
\]

(ii) \[
\lim_{n \to \infty} \sum_{k=0}^{2^n-1} \hat{\mu}^*(k)w_k(x) = 0 \quad \text{uniformly everywhere except on} \quad I.
\]

**Proof.** From the hypothesis, there exist an element of the dyadic group, \( x_0 \), and an integer \( N \) such that \( I = I_N(x_0) \). Set
\[
\hat{\mu}^*(k) = \frac{1}{2^N} \sum_{j=0}^{2^N-1} w_j(x_0)\hat{\mu}(k+j)
\]
for \( k = 0, 1, \ldots \). Since
\[
\sum_{k=0}^{2^n-1} \hat{\mu}^*(k)w_k(x) = \sum_{s=0}^{2^n-N-1} \sum_{k=s2^{N}}^{2^n-1} \hat{\mu}^*(k)w_k(x)
\]
\[
= \sum_{s=0}^{2^n-N-1} \left( \sum_{k=s2^{N}}^{2^n-1} \hat{\mu}^*(s2^{N}+k)w_{s2^{N}+k}(x) \right)
\]
\[
= \sum_{s=0}^{2^n-N-1} \sum_{k=0}^{2^n-1} \left( \frac{1}{2^N} \sum_{j=0}^{2^n-1} w_j(x_0)\hat{\mu}(s2^{N}+k+j)w_{s2^{N}+k}(x) \right)
\]
\[
= \sum_{s=0}^{2^n-N-1} \sum_{j=0}^{2^n-1} \frac{1}{2^N} w_j(x_0) \sum_{k=0}^{2^n-1} \hat{\mu}(s2^{N}+k+j)w_{s2^{N}+k}(x)
\]
\[
= \sum_{s=0}^{2^n-N-1} \sum_{j=0}^{2^n-1} \frac{1}{2^N} w_j(x_0) w_j(x) \sum_{k=0}^{2^n-1} \hat{\mu}(s2^{N}+k+j)w_{s2^{N}+k}(x) \times w_j(x)
\]
\[
= \sum_{s=0}^{2^n-N-1} \frac{1}{2^N} \left( \sum_{j=0}^{2^n-1} w_j(x_0+x) \right) \left\{ \sum_{k=s2^{N}}^{2^n-1} \hat{\mu}(k)w_k(x) \right\}
\]
\[
= \left\{ \frac{1}{2^N} \sum_{j=0}^{2^n-1} w_j(x_0+x) \right\} \left\{ \sum_{k=0}^{2^n-1} \hat{\mu}(k)w_k(x) \right\},
\]
and

\[
\sum_{j=0}^{2^N-1} w_j(x_0 + x) = \begin{cases} 2^N, & \text{for } x \in I_N(x_0), \\ 0, & \text{otherwise}, \end{cases}
\]

we have

\[
\sum_{k=0}^{2^n-1} \hat{\mu}(k)w_k(x) = \begin{cases} \sum_{k=0}^{2^n-1} \hat{\mu}(k)w_k(x), & \text{for } x \in I_N(x_0), \\ 0, & \text{otherwise}. \end{cases}
\]

It is obvious that \( \mu^* \) satisfies the conclusion.

**Lemma 3.** If a Walsh series \( \mu \) satisfies the following conditions:

(i) \[ \liminf_{n \to \infty} \left| \sum_{k=0}^{2^n-1} \hat{\mu}(k)w_k(x) \right| = 0 \quad a.e.; \]

(ii) \[ \sup_n \left| \sum_{k=0}^{2^n-1} \hat{\mu}(k)w_k(x) \right| < \infty \quad \text{everywhere except on a countable set}; \]

(iii) \[ \liminf_{n \to \infty} \left| \frac{1}{2^n} \sum_{k=0}^{2^n-1} \hat{\mu}(k)w_k(x) \right| = 0 \quad \text{everywhere}; \]

then \( \mu \) is the zero series.

Lemma 3 is Theorem 3 in [3].

**Corollary 4.** When \( \mathcal{A} \) satisfies the condition (L), if \( E \) is a closed \( U \)-set for \( \mathcal{A} \) and \( I \) is a dyadic interval which contains \( E \), if a Walsh series \( \mu \in \mathcal{A} \) satisfies the following conditions:

(i) \[ \sup_n \left| \sum_{k=0}^{2^n-1} \hat{\mu}(k)w_k(x) \right| < \infty \quad \text{everywhere on } I \setminus E; \]

(ii) \[ \liminf_{n \to \infty} \left| \sum_{k=0}^{2^n-1} \hat{\mu}(k)w_k(x) \right| = 0 \quad a.e. \text{ on } I; \]

then

\[ \lim_{n \to \infty} \sum_{k=0}^{2^n-1} \hat{\mu}(k)w_k(x) = 0 \quad \text{everywhere on } E. \]

**Proof.** Since \( E \) is a closed set, for \( x_0 \in I \setminus E \), there exists an integer \( N \) such that \( I_N(x_0) \subset I \) and

\[ I_N(x_0) \cap E = \varnothing. \]

Let \( \mu^* \) be a Walsh series which is introduced in Lemma 2. Hence \( \mu^* \) satisfies that

\[ \lim_{n \to \infty} \sum_{k=0}^{2^n-1} \hat{\mu}^*(k)w_k(x) = 0 \quad \text{everywhere on } I_N(x_0). \]
We shall prove that $\mu^*$ satisfies the conditions of Lemma 3. Since $\sum_{k=0}^{2^n-1} \hat{p}(k) w_k(x)$ and $\sum_{k=0}^{2^n-1} \hat{p}^*(k) w_k(x)$ are equiconvergent on $I_N(x_0)$, $\mu^*$ satisfies (5), (i) and (ii) of Lemma 3. From (i) we have

$$\sup_n \left| \sum_{k=0}^{2^n-1} \hat{p}^*(k) w_k(x) \right| < \infty \quad \text{everywhere on } I_N(x_0).$$

On the other hand from (5), (6) holds on $I_N(x_0)$. Hence (6) holds everywhere. From the definition of $\hat{p}^*(k)$ and the hypothesis, we have $\mu^* \in \mathscr{A}$. By Lemma 3, $\mu^*$ is the zero series. Then, we have

$$\lim_{n \to \infty} \sum_{k=0}^{2^n-1} \hat{p}(k) w_k(x) = 0 \quad \text{everywhere on } I \setminus E.$$

Let $\mu^{**}$ be a Walsh series associated with $I$ which is introduced in Lemma 2. Since $\mu^{**} \in \mathscr{A}$ and the $2^n$th partial sums of $\mu^{**}$ and $\mu$ are equiconvergent on $I$, we have

$$\lim_{n \to \infty} \sum_{k=0}^{2^n-1} \hat{p}^{**}(k) w_k(x) = 0 \quad \text{everywhere } I \text{ except on } E.$$

On the other hand, (7) holds on $I^c$. Hence (7) holds everywhere except on $E$. Since $E$ is a $U$-set for $\mathscr{A}$, $\mu^{**}$ is the zero series. Therefore, (7) holds everywhere on $I$. Since the $2^n$th partial sums of $\mu$ and $\mu^{**}$ are equiconvergent, the proof of Corollary 4 is complete.

**Lemma 5.** Let $f_n$, $n = 0, 1, \ldots$, be a function which is continuous on the dyadic group, then the following set

$$N = \left\{ x : \limsup_{n \to \infty} |f_n(x)| = \infty \right\}$$

is empty, countable or of the second category on itself.

The proof is due to [2].

**3. Proof of Theorem 1.** Set $E = \bigcup_{i=1}^{\infty} E_i$. Let $\mu$ satisfy

$$\liminf_{n \to \infty} \left| \sum_{k=0}^{2^n-1} \hat{p}(k) w_k(x) \right| = 0 \quad \text{everywhere except on } E.$$

Each $E_i$ is a $U$-set for $\mathscr{A}$, then from (i) of (L), $E_i$ is of measure zero. Hence $E$ is of measure zero. Consequently $\mu$ satisfies (8) a.e. Set

$$N = \left\{ x : \limsup_{n \to \infty} \left| \sum_{k=0}^{2^n-1} \hat{p}(k) w_k(x) \right| = \infty \right\}.$$

Since $\sum_{k=0}^{2^n-1} \hat{p}(k) w_k(x)$ is a continuous function on the dyadic group, by Lemma 5, three cases arise.

Now we shall assume that $N$ is of the second category on itself. Set $N_i = N \cap E_i$. Then, there exist a dyadic interval $I$ and an integer $i_0$ such that $N \cap I \neq \emptyset$ and $N_{i_0} \cap I$ is dense in $N \cap I$. Since $E_{i_0}$ is closed, we have $N_{i_0} = N \cap E_{i_0}$. We shall prove that

$$N \cap I = E_{i_0} \cap N \cap I \subseteq E_{i_0} \cap I.$$
It is obvious that $N \cap I \supseteq E_{i_0} \cap N \cap I$. If $x \in N \cap I$, then there exists a sequence of elements $\{x_k\}$ such that $x_k \in N_{i_0} \cap I$ and $\lim_{k \to \infty} x_k = x$. Since $x_k \in N_{i_0}$ and $x_k \in I$, we have $x_k \in E_{i_0}$. $E_{i_0}$ is closed, therefore we have $x \in E_{i_0}$. Hence we proved the conclusion. It is obvious that $E_{i_0} \cap I$ is a closed $U$-set for $\mathcal{A}$ and that $E_{i_0} \cap I \subseteq I$. Assume that $x \notin E_{i_0} \cap I$. Then $x \notin N \cap I$. Hence we have

$$\sup_n \left| \sum_{k=0}^{2^n-1} \hat{\mu}(k)w_k(x) \right| < \infty \quad \text{in } I \setminus (E_{i_0} \cap I) \equiv I \setminus E_{i_0}.$$  

By Corollary 4, we have

$$\lim_{n \to \infty} \sum_{k=0}^{2^n-1} \hat{\mu}(k)w_k(x) = 0 \quad \text{everywhere on } I.$$  

Hence we proved that

$$(10) \quad N \cap I = \emptyset.$$  

(10) contradicts the assumption $N \cap I \neq \emptyset$. Therefore we have that $N$ is not of the second category on itself. The proof is complete.

A subset $E$ of the dyadic group is said to be a $U_1$-set for $\mathcal{A}$ if $\mu \in \mathcal{A}$ and

$$(2') \quad \liminf_{n \to \infty} \left| \sum_{k=0}^{2^n-1} \hat{\mu}(k)w_k(x) \right| = 0 \quad \text{everywhere except on } E$$  

imply that $\mu$ is the zero series.

We can prove analogously to Theorem 1 the following theorem.

**Theorem 1'**. When a class of Walsh series $\mathcal{A}$ satisfies the condition (L), if $E_1, E_2, \ldots$ are closed $U_1$-sets for $\mathcal{A}$, then $\bigcup_{k=1}^{\infty} E_k$ is also a $U_1$-set for $\mathcal{A}$.

**References**


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