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RESONANCE AND BIFURCATION OF HIGHER-DIMENSIONAL TORI

DIETRICH FLOCKERZI

ABSTRACT. By means of an example it is shown that a supercritical bifurcation of an invariant 2-torus into an invariant 3-torus prevailing in the case of nonresonance may be replaced by a transcritical bifurcation into a pinched invariant 3-torus in the case of resonance. The connections of these bifurcation phenomena with the properties of the spectrum of the underlying invariant 2-torus are discussed.

1. Statement of the results.

1.1 *Introduction*. In this note we propose a local study of the following smooth 2-parameter system

$$\text{(S)}\qquad \begin{aligned} &\dot{x} = A(y,\alpha)x - \|x\|_2^2 x, \qquad x \in \mathbf{R}^2,\\ &\dot{y}_1 = g(y,\mu,\alpha), \qquad \dot{y}_2 = 1, \qquad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in T^2, \end{aligned}$$

describing the flow of a 4-dimensional system of ordinary differential equations in a small neighborhood of the invariant 2-torus $M^2(\mu,\alpha) = \{x = 0\}$. T^2 denotes the standard 2-torus so that y is defined modulo 2π. For a fixed irrational ω and for relatively prime integers $p \neq 0$ and $q > 0$ we take A and g to be

$$\begin{aligned} &A\colon T^2 \times (-\bar{\alpha}, \bar{\alpha}) \to \mathbf{R}^2 \qquad (\bar{\alpha} > 0 \text{ sufficiently small}),\\ &A(y,\alpha) = \bar{A}(\alpha) + 2\alpha \operatorname{diag}(\cos(qy_1 - py_2)), \qquad \bar{A}(\alpha) = \begin{pmatrix} \alpha & -\omega \\ \omega & \alpha \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} &g\colon T^2 \times [0,\bar{\mu}] \times (-\bar{\alpha}, \bar{\alpha}) \to T^1 \qquad (\bar{\mu} > 0 \text{ sufficiently small}),\\ &g(y,\mu,\alpha) = p/q + \mu + \alpha \sin(qy_1 - py_2). \end{aligned}$$

Thus $\alpha \in (-\bar{\alpha}, \bar{\alpha})$ is to be considered as the bifurcation parameter and $\mu \in [0, \bar{\mu}]$ as an additional parameter varying the rotation vector on the invariant 2-torus. We restrict our attention to positive μ since the case of negative μ can be transformed to this case by changing the sign of y_1 and p. Since $M^2(\mu, 0)$ is a vague attractor for (S) and since the form of the mean value $\bar{A}(\alpha)$ of $A(y,\alpha)$ seems to suggest that $M^2(\mu,\alpha)$ is stable for negative and unstable for positive α, one may expect a supercritical bifurcation of a stable invariant 3-torus $M^3(\mu,\alpha)$ from the 2-torus $M^2(\mu,\alpha)$ at $\alpha = 0$. This is indeed the case for $\mu > 0$ due to the fact that for positive μ the normal portion $\Sigma^N(\mu,\alpha)$ of the spectrum of $M^2(\mu,\alpha)$ (cf. [7]) crosses with α

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from $\mathbf{R}^-$ to $\mathbf{R}^+$. The situation in the resonant case $\mu = 0$ is different in so far as $\Sigma^N(0, \alpha)$ always contains 0 in its interior if α is nonzero. This leads to a two-sided bifurcation of a pinched, stable invariant 3-torus $M^{3-}(0, \alpha)$ at $\alpha = 0$.

1.2 *The bifurcations and the attractors of* (S). To describe our results in more detail we first define the rays

$$L_1^{\pm} = \{(\mu, \alpha) \in (0, \bar{\mu}) \times (-\bar{\alpha}, \bar{\alpha}) \colon \pm\alpha = \mu\},$$
$$L_2^{\pm} = \{(\mu, \alpha) \in (0, \bar{\mu}) \times (-\bar{\alpha}, \bar{\alpha}) \colon \pm\alpha = (2/\sqrt{3})\mu\}.$$

We are interested in a local analysis and thus in the restrictions of these sets to a sufficiently small neighborhood U of $(\mu, \alpha) = (0, 0)$. We denote these restrictions to U by the same symbols. In Figure 1 we have sketched these bifurcation curves $L_i^{\pm}$ and have introduced the regions $\mathrm{I}^{\pm}$, $\mathrm{II}^{\pm}$ and $\mathrm{III}^{\pm}$.

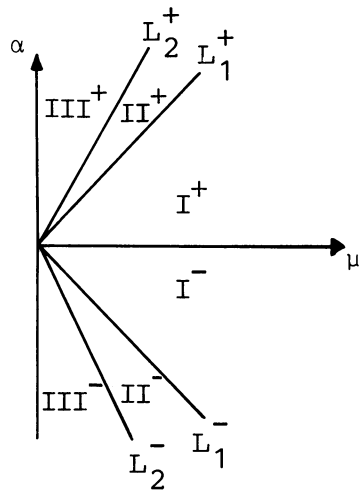

FIGURE 1

In §2 of this note we prove the following changes in the structure of the attractors of system (S):

(1.2.1) The positive μ-axis corresponds to the Hopf-type bifurcation of a unique invariant 3-torus $M^3(\mu, \alpha)$ from $M^2(\mu, \alpha)$. The 2-torus $M^2(\mu, 0)$ is a vague attractor for nonnegative μ. The 3-torus $M^3(\mu, \alpha)$ exists for all $(\mu, \alpha) \in \mathrm{I}^+ \cup L_1^+ \cup \mathrm{II}^+$. In region I^+ it is the only attractor of (S), the 2-torus $M^2(\mu, \alpha)$ is a repellor.

(1.2.2) The ray L_1^+ corresponds to a saddle node bifurcation creating q invariant 2-tori on $M^3(\mu, \alpha)$. (If $M^3(\mu, \alpha)$ is identified with a circle, these 2-tori are saddle nodes.) On L_1^+ and in II^+, the q invariant 2-tori on $M^3(\mu, \alpha)$ are the only attractors of (S). $M^2(\mu, \alpha)$ is still a repellor.

(1.2.3) The ray L_2^+ corresponds to Naimark-Sacker bifurcations causing the pinching of the 3-torus $M^3(\mu, \alpha)$ to $M^{3-}(\mu, \alpha)$. On L_2^+, in III^+ and on the positive α-axis, the attractors of (S) are the q invariant circles along which $M^{3-}(\mu, \alpha)$ coincides with $M^2(\mu, \alpha)$.

(1.2.4) In region I^-, $M^2(\mu, \alpha)$ is the attractor of (S).

(1.2.5) The ray L_1^- corresponds to another saddle node bifurcation creating q invariant circles of saddle node type on $M^2(\mu, \alpha)$. On L_1^-, in II^- and on L_2^-, the only attractors of (S) are the q invariant circles on $M^2(\mu, \alpha)$.

(1.2.6) Finally, the ray L_2^- represents the locus of another Naimark-Sacker bifurcation. In region III^- and on the negative α-axis, there exists a pinched invariant 3-torus $M^{3-}(\mu, \alpha)$ besides the invariant 2-torus $M^2(\mu, \alpha)$. The only attractors of (S) are the q invariant circles along which $M^{3-}(\mu, \alpha)$ coincides with $M^2(\mu, \alpha)$.

1.3 *Spectral properties of* $M^2(\mu, \alpha)$. In §3 we discuss the behavior of the normal and the tangential portions of the spectrum of the invariant 2-torus $M^2(\mu, \alpha)$ in dependence of μ and α. In particular we show that the bifurcation of the pinched 3-torus $M^{3-}(\mu, \alpha)$ into the full 3-torus $M^3(\mu, \alpha)$ (when crossing L_2^+ from III^+ to II^+) and the annihilation of the pinched 3-torus $M^{3-}(\mu, \alpha)$ (when crossing L_2^- from III^- to II^-) is due to:

(1.3.1) for $(\mu, \alpha) \in \mathrm{III}^\pm$, zero is in the interior of the normal spectrum $\Sigma^N(\mu, \alpha)$ of $M^2(\mu, \alpha)$, and

(1.3.2) for $(\mu, \alpha) \in \mathrm{II}^+$ (II^-), the left (right) endpoint of $\Sigma^N(\mu, \alpha)$ has crossed from $\mathbf{R}^-$ to $\mathbf{R}^+$ (or from $\mathbf{R}^+$ to $\mathbf{R}^-$, respectively). The bifurcations on L_2^+ (L_2^-) thus arise because invariant circles on $M^2(\mu, \alpha)$ which are normally attractive in region III^+ (II^-) are normally repulsive in II^+ (III^-). These changes in normal attractivity cause bifurcations of invariant 2-tori from these invariant circles (Naimark-Sacker bifurcations).

2. The bifurcations and the attractors of (S).

2.1 *The reduction to dimension* 2. By introducing polar coordinates (r, θ) for x in (S) and by changing coordinates on the underlying 2-torus via

$$\begin{pmatrix} \psi \\ \phi \end{pmatrix} = \begin{pmatrix} q & -p \\ a & b \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \qquad a, b \in Z, qb + ap = 1,$$

we obtain the following system in cylindrical coordinates:

$$\text{(S1)} \qquad \begin{aligned} &\dot{r} = \alpha(1 + 2\cos\psi)r - r^3, \qquad \dot{\theta} = \omega, \\ &\dot{\psi} = q\mu + q\alpha\sin\psi, \qquad \dot{\phi} = 1/q + a\mu + a\alpha\sin\psi. \end{aligned}$$

The coordinates are chosen as in Figure 2 so that $\{r = 0\}$ represents the 2-torus $M^2(\mu, \alpha)$. The (r, ψ) system is independent of θ and ϕ and can be analyzed

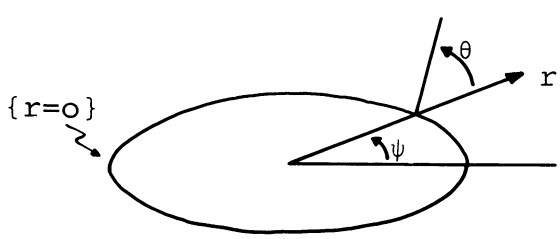

FIGURE 2

separately. For sufficiently small $\bar{\mu}$ and $\bar{\alpha}$, $\dot{\phi}$ is positive and bounded away from 0. Hence one obtains the full 4-dimensional picture for (S1) by adjoining a T^2 to the 2-dimensional one of the (r, ψ)-system. The corresponding figures for the "period map" on the set $\{y_2 = 0 \bmod 2\pi\}$ of the original system (S) are then gotten by cutting along a ray $\psi = \text{const.}$, linearly contracting the (r, ψ)-plane into a wedge of angular width $2\pi/q$ and repeating this contracted figure in each of the q wedges (cf. [**4**]). From now on we frequently deal with the 2-dimensional system

$$\text{(S2)} \qquad \dot{r} = \alpha(1 + 2\cos\psi)r - r^3, \qquad \dot{\psi} = q\mu + q\alpha\sin\psi.$$

Sometimes we interpret (r, ψ) as regular polar coordinates, other times as "blown-up" polar coordinates as shown in Figure 2.

2.2 *The critical points of* (S2). Any trajectory starting at time $t = 0$ in $N = \{(r, \psi): 0 \leqslant r \leqslant r_0\}$ with $r_0 > \sqrt{3}$ remains for all future time in N. For $0 \leqslant |\alpha| < \mu$ there are no critical points in N. For the other values of μ and α one has the following critical points in N:

$$\text{(2.2.1)} \qquad (0, \psi) \quad \text{for } \mu = \alpha = 0, \psi \in T^1;$$

$$\text{(2.2.2)} \qquad (0, \psi_i) \quad \text{for } |\alpha| \geqslant \mu \geqslant 0, (\mu, \alpha) \neq (0, 0),$$

where, for positve μ, $\psi_1 = \psi_1(\mu, \alpha)$ and $\psi_2 = \psi_2(\mu, \alpha)$ are the two solutions of $\dot{\psi} = 0$ with $0 < \psi_1 \leqslant \psi_2 < 2\pi$. For positive (negative) values of α we define $\psi_1(0, \alpha)$ to be π (or 0) and $\psi_2(0, \alpha)$ to be 2π (or π, respectively). Note that on $\{r = 0\}$, $(0, \psi_1)$ is attractive, whereas $(0, \psi_2)$ is repulsive.

$$\text{(2.2.3)} \qquad \begin{aligned} &(r_1, \psi_1) \quad \text{for } \mu \leqslant \alpha \leqslant (2/\sqrt{3})\mu \quad \text{and} \\ &(r_2, \psi_2) \quad \text{for } \alpha \geqslant \mu, -\alpha \geqslant (2/\sqrt{3})\mu, (\mu, \alpha) \neq (0, 0) \\ &\left(r_i = r_i(\mu, \alpha) = \left(\alpha + 2\alpha\cos\psi_i(\mu, \alpha)\right)^{1/2}, i = 1, 2\right). \end{aligned}$$

By considering the 1-dimensional Bernoulli-equation

$$\text{(S3)} \qquad \frac{\partial r}{\partial \psi} = \frac{\alpha(1 + 2\cos\psi)r - r^3}{q\mu + q\alpha\sin\psi}$$

for $0 \leqslant |\alpha| < \mu$, one is led to the conclusion that $\{r = 0\}$ is repulsive for positive and attractive for nonpositive values of α. Since $\dot{\psi}$ is positive, $\{r = 0\}$ behaves like a focus for $0 \leqslant |\alpha| < \mu$ and like a singular node for $\mu = \alpha = 0$. By the Poincaré-Bendixson theory, one therefore obtains a periodic orbit for (S2) in N and hence an invariant 3-torus for (S) as long as α belongs to the interval $(0, \mu)$.

2.3 *The $T^2 \to T^3$ bifurcation on the positive μ-axis.* Following [**3**] we introduce the new bifurcation parameter $\varepsilon > 0$ via $\alpha = \varepsilon\beta$ and use the scaling $r \to \varepsilon r$ in (S1). Then the averaging transformation

$$\bar{r} = r - (2\varepsilon\beta/q\mu)r \cdot \sin\psi, \qquad |2\varepsilon\beta| < q\mu,$$

leads to

$$\dot{\bar{r}} = \varepsilon\bar{r}(\beta - \varepsilon\bar{r}^2) + \varepsilon^2/\mu\,\bar{r}O(\varepsilon^2 + \beta^2),$$

which necessitates the choice $\beta = \pm\varepsilon$. By the further averaging

$$\bar{\psi} = \psi \pm (\varepsilon^2/\mu)\cos\psi, \qquad \bar{\phi} = \phi \pm (a\varepsilon^2/q\mu)\cos\psi$$

with $\varepsilon^2 < \mu$, we generate a weakly coupled system

$$\dot{\bar{r}} = \varepsilon^2 \bar{r}\left(\pm 1 - \bar{r}^2 + O(\varepsilon^2/\mu)\right), \qquad \dot{\theta} = \omega,$$

$$\dot{\bar{\psi}} = q\mu + O(\varepsilon^4/\mu), \qquad \dot{\bar{\phi}} = 1/q + a\mu + O(\varepsilon^4/\mu).$$

Thus for each $\mu_0 > 0$ and $d > 1$, there is a $C > 0$ such that for $0 < \alpha < C\mu^d$, $0 < \mu < \mu_0$, there exists a unique smooth invariant and asymptotically stable 3-torus $M^3(\mu, \alpha)$ for system (S) (cf. [**3**, **2**]). This shows that near, but above, the positive μ-axis system (S2) has a unique periodic orbit $C(\mu, \alpha)$. This orbit is the attractor for (S2).

2.4 *The regions* $I^{\pm}$. We now show that $C(\mu, \alpha)$ is the unique periodic orbit for (S2) not only near the μ-axis but also in all of the region I^+. By (2.2), (2.3) and the global results of [**1**], one expects $C(\mu, \alpha)$ to exist in I^+ with a period tending to infinity as α approaches μ from below. An elementary proof of this fact which also includes the uniqueness of the periodic orbit can be derived by an analysis of (S3). (S3) has a unique attractive 2π-periodic solution for $0 < \alpha < \mu$ and no periodic solution for $-\alpha > \mu > 0$. The period of the corresponding periodic orbit of (S2) tends to infinity as α approaches μ from below. The results of (2.1)–(2.4) now lead to the phase portraits for (S2) that are shown in Figure 3.

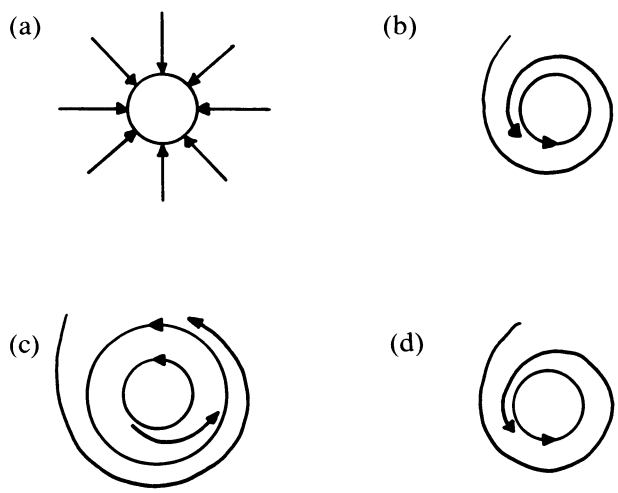

FIGURE 3. Phase portraits of (S2) for (a) $\mu = \alpha = 0$, (b) $\mu > 0, \alpha = 0$, (c) $0 < \alpha < \mu(\mathrm{I}^+)$, and (d) $-\alpha > \mu > 0(\mathrm{I}^-)$.

2.5 *The saddle node bifurcations on* $L_{\mathrm{I}}^{\pm}$ *and the regions* $\mathrm{II}^{\pm}$. For positive α, $\dot{r}$ is negative outside the curve $P_\alpha = \{(r, \psi)\colon r^2 = \alpha + 2\alpha\cos\psi\}$ and positive inside P_α. The critical points (r_i, ψ_i) of (2.2.3) are located on P_α and the integral rays $\psi = \psi_i(\mu, \alpha)$ (cf. (2.2.2)). For $\alpha = \mu$ they coincide, for $\alpha \in (\mu, (2/\sqrt{3})\mu)$ they are different and have a positive first component. As α approaches $(2/\sqrt{3})\mu$ from

below, (r_1, ψ_1) tends to $(0, (4/3)\pi)$ and (r_2, ψ_2) to $(\sqrt{2\alpha}, (5/3)\pi)$. See Figure 4. In our analysis we now interpret r and ψ in (S2) as regular polar coordinates and proceed to determine the type of each critical point. It will turn out that (r_1, ψ_1) is a stable node and (r_2, ψ_2) a saddle point for all (μ, α) in region II^+. On L_1^+ these two points coincide and form a saddle node.

The eigenvalues of the linearization at (r_i, ψ_i), $i = 1, 2$, are given by

$$\lambda_i^N(\mu, \alpha) = -2r_i^2(\mu, \alpha), \qquad \lambda_i^T(\mu, \alpha) = \alpha q \cos \psi_i(\mu, \alpha).$$

Associated eigenvectors with respect to the basis given by the r- and the ψ-directions are

$$v_i^N(\mu, \alpha) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad v_i^T(\mu, \alpha) = \begin{pmatrix} 2r_i \sin \psi_i(\mu, \alpha) \\ -2 + (q + 4)\cos \psi_i(\mu, \alpha) \end{pmatrix}.$$

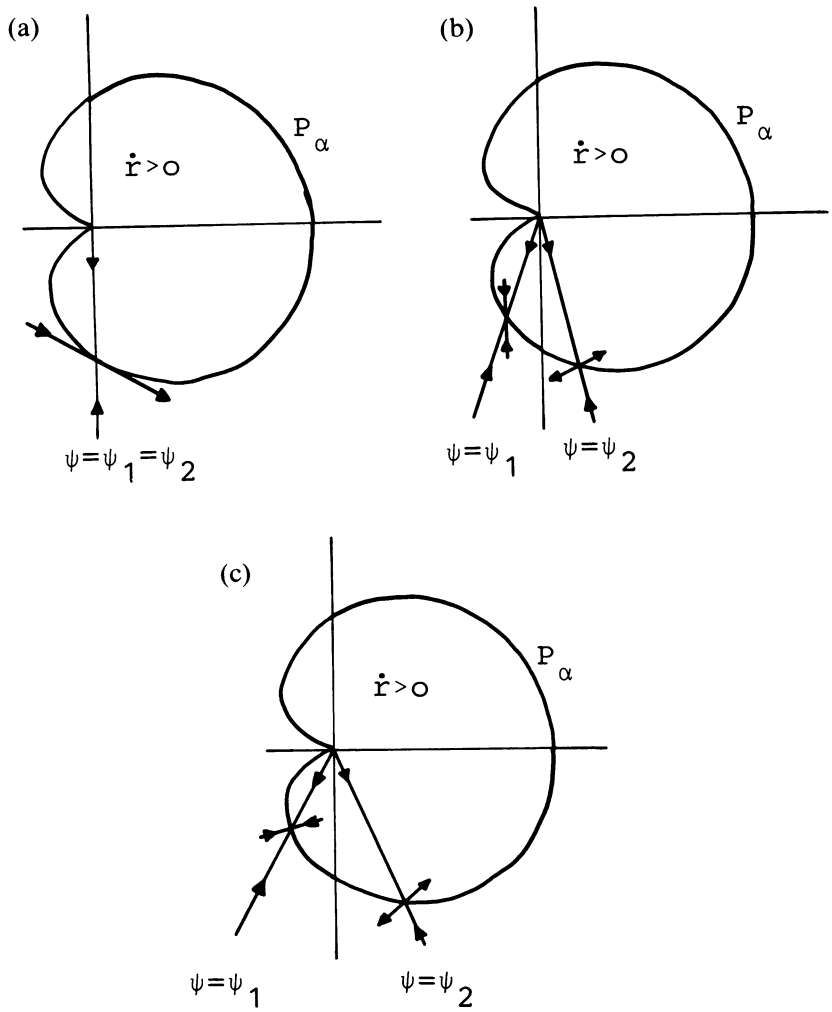

FIGURE 4. Linearizations of (S2) at (r_i, ψ_i), $i = 1, 2$.

In region II^+, λ_i^N is different from λ_i^T except for (μ, α) belonging to the ray

$$L_*^+ = \left\{(\mu, \alpha): \alpha = \left(1 - (2/(q+4))^2\right)^{-1/2}\mu > 0\right\},$$

on which v_1^N and v_1^T are collinear. If we denote the angle ψ_1, defining L_*^+ by $\psi_1^* \in (\pi, (3/2)\pi)$, the critical point (r_1, ψ_1^*) is a degenerate node. In Figure 4 we have sketched the curve P_α and the integral rays $\dot{\psi} = 0$, and have indicated the eigenspaces generated by v_i^T for (μ, α) on L_1^+ (Figure 4(a)), for (μ, α) in II^+, below L_*^+ (Figure 4(b)), and above L_*^+ (Figure 4(c)).

Because of the results in (2.1) and (2.5), the two branches of the unstable manifold of (r_2, ψ_2) must join the critical point (r_1, ψ_1) in the region $\{(r, \psi): 0 < r < r_0\}$,

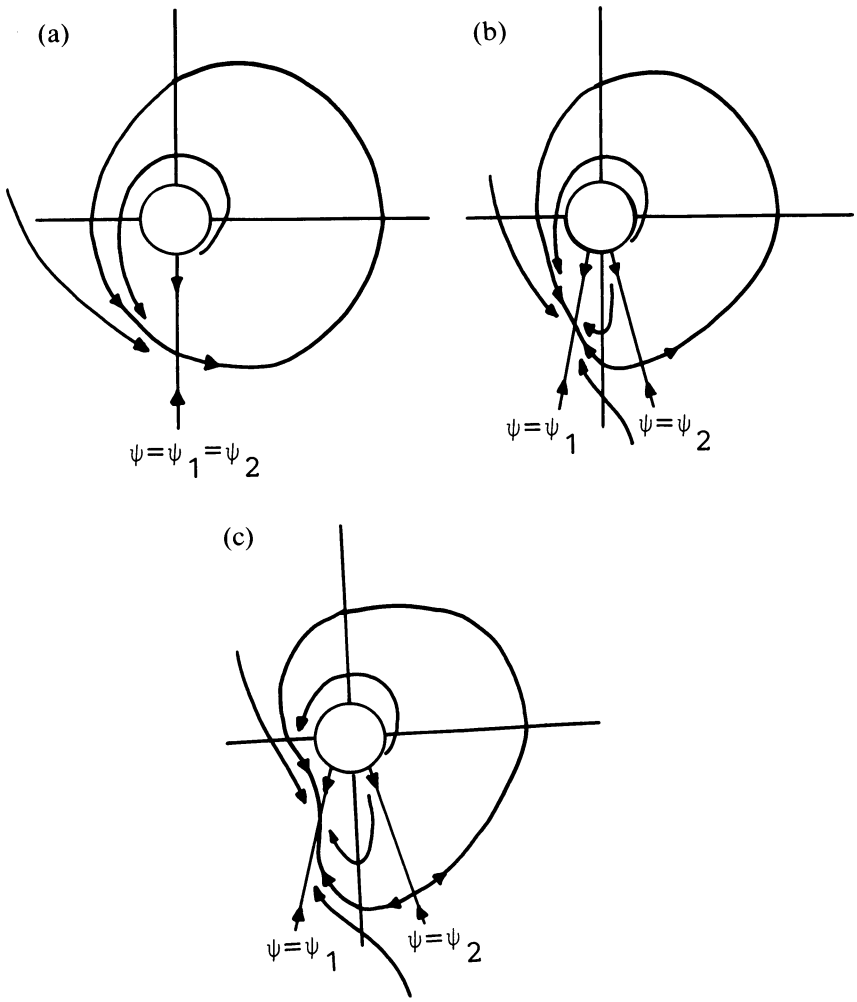

FIGURE 5. Phase portraits of (S2) (a) on L_1^+, (b) in II^+ with $\psi_1 > \psi_1^*$ and (c) in II^+ with $\psi_1 < \psi_1^*$.

giving a homoclinic orbit in the situation of Figure 4(a) and two heteroclinic orbits for Figure 4(b), (c). In blown-up coordinates we thus have established the following: On L_1^+ and in II^+ one has a unique invariant circle $C(\mu, \alpha)$ in $\{(r, \psi): 0 < r < r_0\}$. The attractor of (S2) is the point $(r_1(\mu, \alpha), \psi_1(\mu, \alpha))$ on this circle. The circle $\{r = 0\}$ is repulsive. See Figure 5. For a general discussion of such a saddle node bifurcation, we refer to [**2**, pp. 360–362].

The problem is much easier to handle for negative values of α. A similar analysis shows that the circle $\{r = 0\}$ is attractive on L_1^-, in II^-, and on L_2^-. The attractor of (S2) is the critical point $(0, \psi_1(\mu, \alpha))$. See Figure 6. If we translate the results of 2.4 and 2.5 into the context of system (S) we have shown points (1.2.1), (1.2.2), (1.2.4) and (1.2.5).

2.6 *The Naimark-Sacker bifurcations on $L_2^\pm$ and the regions* $\mathrm{III}^\pm$. On L_2^+ the critical point (r_1, ψ_1) undergoes a saddle node bifurcation with the critical point $(0, \psi_1)$ so that in region III^+ and on the positive α-axis, the unstable manifold of (r_2, ψ_2) joins the critical point $(0, \psi_1)$ on the circle $\{r = 0\}$. The invariant circle $C(\mu, \alpha)$ now develops a cusp at $(0, \psi_1)$. See Figure 7.

We would like to remark that this saddle node bifurcation for the 2-dimensional system (S2) is, in fact, a Naimark-Sacker bifurcation for the 4-dimensional system

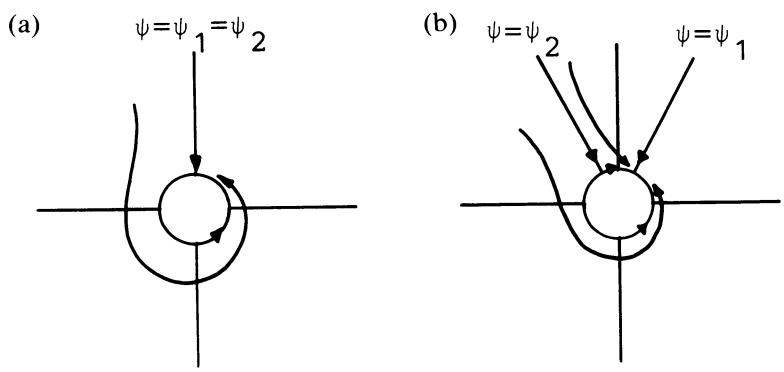

FIGURE 6. Phase portraits of (S2) (a) on L_1^-, and (b) in II^- and on L_2^-.

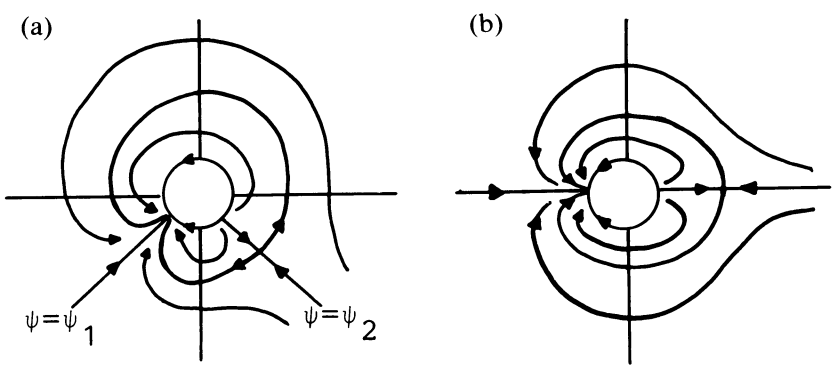

FIGURE 7. Phase portraits of (S2) (a) in III^+, and (b) for $\mu = 0$, $\alpha > 0$.

(S) or (S1). This follows immediately from an analysis of (S1). For ψ_1 close to $(4/3)\pi$ and $\psi_1 > (4/3)\pi$, the circle$\{(\psi, \phi)\colon \psi = \psi_1\}$ is repulsive; for $\psi_1 \leqslant (4/3)\pi$, it is attractive. This leads to the contraction of the 2-torus $\{(r, \theta, \psi, \phi)\colon r = r_1, \psi = \psi_1\}$ to the circle $\{(\psi, \phi)\colon \psi = \psi_1\}$ when ψ_1 decreases through $(4/3)\pi$.

For $-\alpha > (2/\sqrt{3})\mu \geqslant 0$ the critical points of (S2) are the stable node $(0, \psi_1)$ and the saddle points $(0, \psi_2)$ and (r_2, ψ_2). On L_2^- the critical points $(0, \psi_2)$ and (r_2, ψ_2) coincide and form a saddle node; in III^- and on the negative α-axis r_2 is positive. As above, the unstable manifold of (r_2, ψ_2) joins the node $(0, \psi_1)$. See Figure 8. Again, the translation into the 4-dimensional context of system (S) yields the claims of (1.2.3) and (1.2.6).

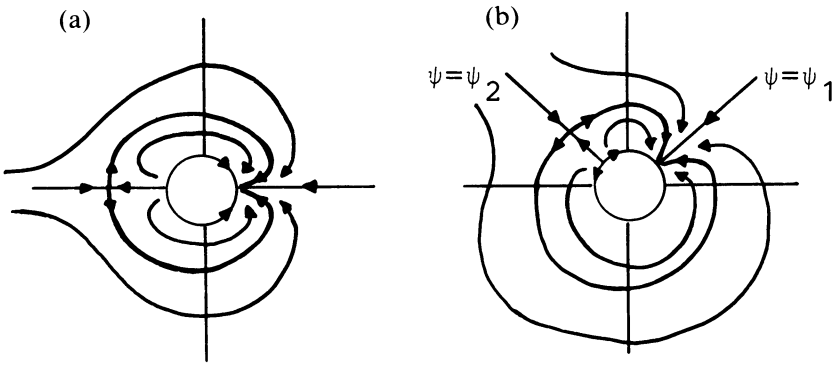

FIGURE 8. Phase portraits for (S2) (a) for $\mu = 0$, $\alpha < 0$, and (b) in III^-.

3. Spectral properties of $M^2(\mu, \alpha)$.

3.1 *Preliminaries.* We start by repeating the notions of the spectrum of an invariant manifold that are relevant for our analysis. For more details we refer to [**6** and **7**]. We take ψ and ϕ as the coordinates on the invariant 2-torus $M^2(\mu, \alpha)$ and denote the flow on $M^2(\mu, \alpha)$ generated by

$$\dot{\xi} = \begin{pmatrix} q\mu + q\alpha \sin\psi \\ \dfrac{1}{q} + a\mu + a\alpha \sin\psi \end{pmatrix}, \qquad \xi = \begin{pmatrix} \psi \\ \phi \end{pmatrix} \in T^2, \tag{1}$$

by $\sigma(\xi, \mu, \alpha, t) = \binom{\Psi}{\Phi}(\xi, \mu, \alpha, t)$, $\sigma(\xi, \mu, \alpha, 0) = \xi$. We choose $\bar{\mu} > 0$ and $\bar{\alpha} > 0$ so that $\dot{\phi}$ is positive for all $(\xi, \mu, \alpha) \in T^2 \times [0, \bar{\mu}] \times [-\bar{\alpha}, \bar{\alpha}]$. Then we define the λ-shifted variational equations along σ for the normal and the tangential directions by

$(2)_\lambda$

$$\dot{u} = (A(\Psi(\xi, \mu, \alpha, t), \alpha) - \lambda I)u, \qquad A(\psi, \alpha) = \begin{pmatrix} \alpha + 2\alpha\cos\psi & -\omega \\ \omega & \alpha + 2\alpha\cos\psi \end{pmatrix},$$

$$(3)_\lambda \qquad \dot{v} = (B(\Psi(\xi, \mu, \alpha, t), \alpha) - \lambda I)v, \qquad B(\psi, \alpha) = \begin{pmatrix} q\alpha\cos\psi & 0 \\ a\alpha\cos\psi & 0 \end{pmatrix},$$

respectively. For fixed $\xi \in T^2$ the normal portion $\Sigma^N(\xi, \mu, \alpha)$ and the tangential portion $\Sigma^T(\xi, \mu, \alpha)$ along the solution $\sigma(\xi, \mu, \alpha, t)$ of (1) are then

$$\Sigma^N(\xi, \mu, \alpha) = \{\lambda \in \mathbf{R}\colon (2)_\lambda \text{ does not admit an exponential dichotomy}\},$$
$$\Sigma^T(\xi, \mu, \alpha) = \{\lambda \in \mathbf{R}\colon (3)_\lambda \text{ does not admit an exponential dichotomy}\}.$$

The normal portion $\Sigma^N(\mu, \alpha)$ and the tangential portion $\Sigma^T(\mu, \alpha)$ of the spectrum of the 2-torus $M^2(\mu, \alpha)$ are then the unions

$$\Sigma^N(\mu, \alpha) = \bigcup_{\xi \in T^2} \Sigma^N(\xi, \mu, \alpha), \qquad \Sigma^T(\mu, \alpha) = \bigcup_{\xi \in T^2} \Sigma^T(\xi, \mu, \alpha).$$

3.2 *The normal spectrum* $\Sigma^N(\mu, \alpha)$. For $0 \leqslant |\alpha| < \mu$ the normal spectrum $\Sigma^N(\mu, \alpha)$ is equal to $\{\alpha\}$ and the associated spectral subbundle is 2-dimensional. This follows easily from $(2)_\lambda$. We now take the case $|\alpha| > \mu$. In (2.2) we have determined two invariant circles $\Omega_i = \Omega_i(\mu, \alpha) = \{\psi_i(\mu, \alpha)\} \times T^1$, $i = 1, 2$, for (1). Along the solutions $\sigma(\psi_i, \phi, \mu, \alpha, t)$ equation $(2)_\lambda$ is given by

$$\dot{u} = \left(\begin{pmatrix} \alpha(1 \mp 2c) & -\omega \\ \omega & \alpha(1 \mp 2c) \end{pmatrix} - \lambda I\right) u,$$

where we have taken c to be $c(\mu, \alpha) = \cos \psi_2(\mu, \alpha) = \sqrt{1 - (\mu/\alpha)^2}$. The upper (lower) sign refers to ψ_1 (ψ_2, respectively). For any solution of (1) with initial value $\xi = (\psi, \phi)$, $\psi_1 \neq \psi \neq \psi_2$, the positive limit set is Ω_1, the negative limit set Ω_2. Because of Lemma 1 of [**6**] we can conclude that the points $\alpha(1 \mp 2c)$ belong to $\Sigma^N(\xi, \mu, \alpha)$. Since the dimensions of the stable subbundle over Ω_1 and the unstable over Ω_2 are both 2, Theorem 3.2 of [**5**] implies that the interval with $\alpha(1 \mp 2c)$ as endpoints belongs to $\Sigma^N(\xi, \mu, \alpha)$. For any λ outside this interval $(2)_\lambda$ admits an exponential dichotomy. In summarizing we can say that $\Sigma^N(\mu, \alpha)$ is given by $\{\alpha\}$ for $0 \leqslant |\alpha| \leqslant \mu$ and by

$$\Sigma^N(\mu, \alpha) = \begin{cases} [\alpha(1 - 2c(\mu, \alpha)), \alpha(1 + 2c(\mu, \alpha))] & \text{for } \alpha > \mu \geqslant 0, \\ [\alpha(1 + 2c(\mu, \alpha)), \alpha(1 - 2c(\mu, \alpha))] & \text{for } -\alpha > \mu \geqslant 0, \end{cases}$$

and that the associated spectral subbundle is 2-dimensional (cf. [**5, 6**]).

3.3 *The tangential spectrum* $\Sigma^T(\mu, \alpha)$. For $0 \leqslant |\alpha| < \mu$ the 2-torus $M^2(\mu, \alpha)$ is minimal and the only bounded solution of $(3)_\lambda$, $\lambda \neq 0$, is the trivial solution. For $\lambda = 0$ all solutions of $(3)_\lambda$ are bounded. Therefore the tangential spectrum $\Sigma^T(\mu, \alpha)$ is equal to $\{0\}$. For $|\alpha| > \mu \geqslant 0$ there are two invariant circles $\Omega_i(\mu, \alpha)$ on $M^2(\mu, \alpha)$. Again, for any solution of (1) with initial value $\xi = (\psi, \phi)$, $\psi_1 \neq \psi \neq \psi_2$, the positive limit set is Ω_1, the negative Ω_2. The same arguments as in (3.2) lead to

$$\Sigma^T(\mu, \alpha) = \left[-q|\alpha|c(\mu, \alpha), q|\alpha|c(\mu, \alpha)\right] \quad \text{for } |\alpha| > \mu \geqslant 0.$$

3.4 *Consequences.* (1) From (3.2) it follows that for $\mu = 0$, $\alpha \neq 0$ the normal spectrum is an interval containing 0 in its interior. For any fixed positive μ, $\Sigma^N(\mu, \alpha)$ is in $\mathbf{R}^-$ for $-(2/\sqrt{3})\mu < \alpha < 0$, reduces to $\{0\}$ for $\alpha = 0$ and is in $\mathbf{R}^+$ for $0 < \alpha < (2/\sqrt{3})\mu$. Thus the underlying invariant 2-torus $M^2(\mu, \alpha)$ is normally attractive for $-(2/\sqrt{3})\mu < \alpha < 0$ and normally repulsive for $0 < \alpha < (2/\sqrt{3})\mu$. As

α increases (decreases) through $(2/\sqrt{3})\mu$ $(-(2/\sqrt{3})\mu)$ the left (right) endpoint of $\Sigma^N(\mu, \alpha)$ crosses through 0 again. This proves (1.3.1) and (1.3.2).

(2) The spectra $\Sigma^N(\mu, \alpha)$ and $\Sigma^T(\mu, \alpha)$ are disjoint for

$$0 < |\alpha| < \left(1 - (1/(q+2))^2\right)^{-1/2}\mu$$

and $M^2(\mu, \alpha)$ is normally hyperbolic for these (μ, α). Note that the set of (μ, α) defined by the above inequality contains the regions I^+ and I^- and parts of II^+ and II^-.

(3) It is of interest to note that for $\alpha > (2/\sqrt{3})\mu$, the left endpoint of $\Sigma^T(\mu, \alpha)$ is always smaller than the one of $\Sigma^N(\mu, \alpha)$ so that the tangential flow at the attractor on $M^2(\mu, \alpha)$ is faster than the normal flow. That this leads to a cusp of the pinched 3-torus has been shown in (2.6). For $\alpha < -(2/\sqrt{3})\mu$ the same holds for $q \geqslant 4$. For $q = 2$ or $q = 3$ (and $\mu > 0$) the pinched 3-torus is flat where it is pinched.

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Mathematisches Institut, Universität Würzburg, D-8700 Würzburg, Federal Republic of Germany