RESONANCE AND BIFURCATION OF HIGHER-DIMENSIONAL TORI

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Abstract. By means of an example it is shown that a supercritical bifurcation of an invariant 2-torus into an invariant 3-torus prevailing in the case of nonresonance may be replaced by a transcritical bifurcation into a pinched invariant 3-torus in the case of resonance. The connections of these bifurcation phenomena with the properties of the spectrum of the underlying invariant 2-torus are discussed.

1. Statement of the results.

1.1 Introduction. In this note we propose a local study of the following smooth 2-parameter system

$$\begin{align*}
\dot{x} &= A(y, \alpha)x - \|x\|^2x, \quad x \in \mathbb{R}^2, \\
\dot{y}_1 &= g(y, \mu, \alpha), \quad \dot{y}_2 = 1, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in T^2,
\end{align*}$$

(S)

describing the flow of a 4-dimensional system of ordinary differential equations in a small neighborhood of the invariant 2-torus $M_2(\mu, \alpha) = \{x = 0\}$. $T^2$ denotes the standard 2-torus so that $y$ is defined modulo $2\pi$. For a fixed irrational $\omega$ and for relatively prime integers $p \neq 0$ and $q > 0$ we take $A$ and $g$ to be

$$A : T^2 \times (-\bar{\alpha}, \bar{\alpha}) \to \mathbb{R}^2 \quad (\bar{\alpha} > 0 \text{ sufficiently small}),$$

$$A(y, \alpha) = \bar{A}(\alpha) + 2\alpha \text{diag}(\cos(qy_1 - py_2)), \quad \bar{A}(\alpha) = \begin{pmatrix} \alpha & -\omega \\ \omega & \alpha \end{pmatrix},$$

and

$$g : T^2 \times [0, \bar{\mu}] \times (-\bar{\alpha}, \bar{\alpha}) \to T^1 \quad (\bar{\mu} > 0 \text{ sufficiently small}),$$

$$g(y, \mu, \alpha) = \frac{p}{q} + \mu + \alpha \sin(qy_1 - py_2).$$

Thus $\alpha \in (-\bar{\alpha}, \bar{\alpha})$ is to be considered as the bifurcation parameter and $\mu \in [0, \bar{\mu}]$ as an additional parameter varying the rotation vector on the invariant 2-torus. We restrict our attention to positive $\mu$ since the case of negative $\mu$ can be transformed to this case by changing the sign of $y_1$ and $p$. Since $M_2^2(\mu, 0)$ is a vague attractor for (S) and since the form of the mean value $\bar{A}(\alpha)$ of $A(y, \alpha)$ seems to suggest that $M_2^2(\mu, \alpha)$ is stable for negative and unstable for positive $\alpha$, one may expect a supercritical bifurcation of a stable invariant 3-torus $M_3^1(\mu, \alpha)$ from the 2-torus $M_2^2(\mu, \alpha)$ at $\alpha = 0$. This is indeed the case for $\mu > 0$ due to the fact that for positive $\mu$ the normal portion $\Sigma^N(\mu, \alpha)$ of the spectrum of $M_2^2(\mu, \alpha)$ (cf. [7]) crosses with $\alpha$.
from $\mathbb{R}^-$ to $\mathbb{R}^+$. The situation in the resonant case $\mu = 0$ is different in so far as $\Sigma^N(0, \alpha)$ always contains 0 in its interior if $\alpha$ is nonzero. This leads to a two-sided bifurcation of a pinched, stable invariant 3-torus $M^3(0, \alpha)$ at $\alpha = 0$.

1.2 The bifurcations and the attractors of (S). To describe our results in more detail we first define the rays

$$L_1^\pm = \{(\mu, \alpha) \in (0, \mu) \times (-\bar{\alpha}, \bar{\alpha}) : \pm \alpha = \mu\},$$

$$L_2^\pm = \{(\mu, \alpha) \in (0, \bar{\mu}) \times (-\bar{\alpha}, \bar{\alpha}) : \pm \alpha = (2/\sqrt{3})\mu\}.$$  

We are interested in a local analysis and thus in the restrictions of these sets to a sufficiently small neighborhood $U$ of $(\mu, \alpha) = (0,0)$. We denote these restrictions to $U$ by the same symbols. In Figure 1 we have sketched these bifurcation curves $L_i^\pm$ and have introduced the regions $I^\pm$, $II^\pm$ and $III^\pm$.

In §2 of this note we prove the following changes in the structure of the attractors of system (S):

(1.2.1) The positive $\mu$-axis corresponds to the Hopf-type bifurcation of a unique invariant 3-torus $M^3(\mu, \alpha)$ from $M^2(\mu, \alpha)$. The 2-torus $M^2(\mu, 0)$ is a vague attractor for nonnegative $\mu$. The 3-torus $M^3(\mu, \alpha)$ exists for all $(\mu, \alpha) \in I^+ \cup L_1^+ \cup II^+$. In region $I^+$ it is the only attractor of (S), the 2-torus $M^2(\mu, \alpha)$ is a repellor.

(1.2.2) The ray $L_1^+$ corresponds to a saddle node bifurcation creating $q$ invariant 2-tori on $M^3(\mu, \alpha)$. (If $M^3(\mu, \alpha)$ is identified with a circle, these 2-tori are saddle nodes.) On $L_1^+$ and in $II^+$, the $q$ invariant 2-tori on $M^3(\mu, \alpha)$ are the only attractors of (S). $M^2(\mu, \alpha)$ is still a repellor.

(1.2.3) The ray $L_2^+$ corresponds to Naimark-Sacker bifurcations causing the pinching of the 3-torus $M^3(\mu, \alpha)$ to $M^3(\mu, \alpha)$. On $L_2^+$, in $III^+$ and on the positive $\alpha$-axis, the attractors of (S) are the $q$ invariant circles along which $M^3(\mu, \alpha)$ coincides with $M^2(\mu, \alpha)$.

(1.2.4) In region $I^-$, $M^2(\mu, \alpha)$ is the attractor of (S).
(1.2.5) The ray $L_1^-$ corresponds to another saddle node bifurcation creating $q$ invariant circles of saddle node type on $M^2(\mu, \alpha)$. On $L_1^-$, in $\Pi^-$ and on $L_2^-$, the only attractors of (S) are the $q$ invariant circles on $M^2(\mu, \alpha)$.

(1.2.6) Finally, the ray $L_2^-$ represents the locus of another Naimark-Sacker bifurcation. In region $\Pi^-$ and on the negative $\alpha$-axis, there exists a pinched invariant 3-torus $M^3(\mu, \alpha)$ besides the invariant 2-torus $M^2(\mu, \alpha)$. The only attractors of (S) are the $q$ invariant circles along which $M^3(\mu, \alpha)$ coincides with $M^2(\mu, \alpha)$.

1.3 Spectral properties of $M^2(\mu, \alpha)$. In §3 we discuss the behavior of the normal and the tangential portions of the spectrum of the invariant 2-torus $M^2(\mu, \alpha)$ in dependence of $\mu$ and $\alpha$. In particular we show that the bifurcation of the pinched 3-torus $M^3(\mu, \alpha)$ into the full 3-torus $M^3(\mu, \alpha)$ (when crossing $L_2^+$ from $\Pi^+$ to $\Pi^+$) and the annihilation of the pinched 3-torus $M^3(\mu, \alpha)$ (when crossing $L_2^-$ from $\Pi^-$ to $\Pi^-$) is due to:

(1.3.1) for $(\mu, \alpha) \in \Pi^\pm$, zero is in the interior of the normal spectrum $\Sigma^N(\mu, \alpha)$ of $M^2(\mu, \alpha)$, and

(1.3.2) for $(\mu, \alpha) \in \Pi^+ (\Pi^-)$, the left (right) endpoint of $\Sigma^N(\mu, \alpha)$ has crossed from $\mathbb{R}^-$ to $\mathbb{R}^+$ (or from $\mathbb{R}^+$ to $\mathbb{R}^-$, respectively). The bifurcations on $L_2^+$ ($L_2^-$) thus arise because invariant circles on $M^2(\mu, \alpha)$ which are normally attractive in region $\Pi^+$ ($\Pi^-$) are normally repulsive in $\Pi^+$ ($\Pi^-$). These changes in normal attractivity cause bifurcations of invariant 2-tori from these invariant circles (Naimark-Sacker bifurcations).

2. The bifurcations and the attractors of (S).

2.1 The reduction to dimension 2. By introducing polar coordinates $(r, \theta)$ for $x$ in (S) and by changing coordinates on the underlying 2-torus via

$$\begin{pmatrix} \psi \\ \phi \end{pmatrix} = \begin{pmatrix} q & -p \\ a & b \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad a, b \in \mathbb{Z}, \quad qb + ap = 1,$$

we obtain the following system in cylindrical coordinates:

$$\begin{align*}
\dot{r} &= \alpha(1 + 2 \cos \psi)r - r^3, \\
\dot{\theta} &= \omega, \\
\dot{\psi} &= q\mu + q\alpha \sin \psi, \\
\dot{\phi} &= 1/q + a\mu + a\alpha \sin \psi.
\end{align*}$$

The coordinates are chosen as in Figure 2 so that $\{r = 0\}$ represents the 2-torus $M^2(\mu, \alpha)$. The $(r, \psi)$ system is independent of $\theta$ and $\phi$ and can be analyzed.

![Figure 2](http://www.ams.org/journal-terms-of-use)
separately. For sufficiently small \( \bar{\mu} \) and \( \bar{\alpha} \), \( \phi \) is positive and bounded away from 0. Hence one obtains the full 4-dimensional picture for (S1) by adjoining a \( T^2 \) to the 2-dimensional one of the \((r, \psi)\)-system. The corresponding figures for the “period map” on the set \( \{ \psi_2 = 0 \mod 2\pi \} \) of the original system (S) are then gotten by cutting along a ray \( \psi = \text{const.} \), linearly contracting the \((r, \psi)\)-plane into a wedge of angular width \( 2\pi/q \) and repeating this contracted figure in each of the \( q \) wedges (cf. [4]). From now on we frequently deal with the 2-dimensional system

\[ \dot{r} = \alpha(1 + 2\cos \psi)r - r^3, \quad \dot{\psi} = q\mu + q\alpha \sin \psi. \]

Sometimes we interpret \((r, \psi)\) as regular polar coordinates, other times as “blown-up” polar coordinates as shown in Figure 2.

2.2 The critical points of (S2). Any trajectory starting at time \( t = 0 \) in \( N = \{(r, \psi): 0 \leq r < r_0 \} \) with \( r_0 > \sqrt{3} \) remains for all future time in \( N \). For \( 0 < |\alpha| < \mu \) there are no critical points in \( N \). For the other values of \( \mu \) and \( \alpha \) one has the following critical points in \( N \):

\[
\begin{align*}
& (0, \psi) \quad \text{for } \mu = \alpha = 0, \psi \in T^1; \\
& (0, \psi_i) \quad \text{for } |\alpha| \geq \mu \geq 0, (\mu, \alpha) \neq (0, 0),
\end{align*}
\]

where, for positive \( \mu \), \( \psi_1 = \psi_i(\mu, \alpha) \) and \( \psi_2 = \psi_2(\mu, \alpha) \) are the two solutions of \( \dot{\psi} = 0 \) with \( 0 < \psi_1 \leq \psi_2 < 2\pi \). For positive (negative) values of \( \alpha \) we define \( \psi_1(0, \alpha) \) to be \( \pi \) (or 0) and \( \psi_2(0, \alpha) \) to be \( 2\pi \) (or \( \pi \), respectively). Note that on \( \{r = 0\}, (0, \psi_1) \) is attractive, whereas \((0, \psi_2)\) is repulsive.

\[
\begin{align*}
& (r_1, \psi_1) \quad \text{for } \mu \leq \alpha \leq (2/\sqrt{3})\mu \quad \text{and} \\
& (r_2, \psi_2) \quad \text{for } \alpha > \mu, -\alpha > (2/\sqrt{3})\mu, (\mu, \alpha) \neq (0, 0) \\
& (r_i = r_i(\mu, \alpha) = (\alpha + 2\alpha \cos \psi_i(\mu, \alpha))^{1/2}, i = 1, 2).
\end{align*}
\]

By considering the 1-dimensional Bernoulli-equation

\[ \frac{\partial r}{\partial \psi} = \frac{\alpha(1 + 2\cos \psi)r - r^3}{q\mu + q\alpha \sin \psi} \]

for \( 0 \leq |\alpha| < \mu \), one is led to the conclusion that \( \{r = 0\} \) is repulsive for positive and attractive for nonpositive values of \( \alpha \). Since \( \psi \) is positive, \( \{r = 0\} \) behaves like a focus for \( 0 \leq |\alpha| < \mu \) and like a singular node for \( \mu = \alpha = 0 \). By the Poincaré-Bendixson theory, one therefore obtains a periodic orbit for (S2) in \( N \) and hence an invariant 3-torus for (S) as long as \( \alpha \) belongs to the interval \( (0, \mu) \).

2.3 The \( T^2 \to T^3 \) bifurcation on the positive \( \mu \)-axis. Following [3] we introduce the new bifurcation parameter \( \epsilon > 0 \) via \( \alpha = \epsilon \beta \) and use the scaling \( r \to \epsilon r \) in (S1). Then the averaging transformation

\[ \tilde{r} = r - (2\epsilon \beta / q\mu) r \cdot \sin \psi, \quad |2\epsilon \beta| < q\mu, \]

leads to

\[ \tilde{r} = \epsilon \tilde{r}(\beta - \epsilon \tilde{r}^2) + \epsilon^2 / \mu \tilde{r}O(\epsilon^2 + \beta^2), \]

which necessitates the choice \( \beta = \pm \epsilon \). By the further averaging

\[ \tilde{\psi} = \psi \pm (\epsilon^2 / \mu) \cos \psi, \quad \tilde{\phi} = \phi \pm (a \epsilon^2 / q\mu) \cos \psi \]
with $\epsilon^2 < \mu$, we generate a weakly coupled system

$$
\begin{aligned}
\dot{r} &= \epsilon^2 r \left( \pm 1 - r^2 + O(\epsilon^2/\mu) \right), \\
\dot{\psi} &= q\mu + O(\epsilon^4/\mu), \\
\dot{\phi} &= 1/q + a\mu + O(\epsilon^4/\mu).
\end{aligned}
$$

Thus for each $\mu_0 > 0$ and $d > 1$, there is a $C > 0$ such that for $0 < \alpha < C\mu^d$, $0 < \mu < \mu_0$, there exists a unique smooth invariant and asymptotically stable 3-torus $M^3(\mu, \alpha)$ for system (S) (cf. [3, 2]). This shows that near, but above, the positive \(\mu\)-axis system (S2) has a unique periodic orbit $C(\mu, \alpha)$. This orbit is the attractor for (S2).

2.4 The regions $I^\pm$. We now show that $C(\mu, \alpha)$ is the unique periodic orbit for (S2) not only near the $\mu$-axis but also in all of the region $I^\pm$. By (2.2), (2.3) and the global results of [1], one expects $C(\mu, \alpha)$ to exist in $I^+$ with a period tending to infinity as $\alpha$ approaches $\mu$ from below. An elementary proof of this fact which also includes the uniqueness of the periodic orbit can be derived by an analysis of (S3). (S3) has a unique attractive $2\pi$-periodic solution for $0 < \alpha < \mu$ and no periodic solution for $-\alpha > \mu > 0$. The period of the corresponding periodic orbit of (S2) tends to infinity as $\alpha$ approaches $\mu$ from below. The results of (2.1)–(2.4) now lead to the phase portraits for (S2) that are shown in Figure 3.

![Phase portraits of (S2)](image)

**Figure 3.** Phase portraits of (S2) for (a) $\mu = \alpha = 0$, (b) $\mu > 0, \alpha = 0$, (c) $0 < \alpha < \mu(1^+)$, and (d) $-\alpha > \mu > 0(1^-)$.

2.5 The saddle node bifurcations on $L_f^\pm$ and the regions $II^\pm$. For positive $\alpha$, $\dot{r}$ is negative outside the curve $P_\alpha = \{(r, \psi): r^2 = \alpha + 2\alpha \cos \psi\}$ and positive inside $P_\alpha$. The critical points $(r_i, \psi_i)$ of (2.2.3) are located on $P_\alpha$ and the integral rays $\psi = \psi_i(\mu, \alpha)$ (cf. (2.2.2)). For $\alpha = \mu$ they coincide, for $\alpha \in (\mu, (2/\sqrt{3})\mu)$ they are different and have a positive first component. As $\alpha$ approaches $(2/\sqrt{3})\mu$ from...
below, \((r_1, \psi_1)\) tends to \((0, (4/3)\pi)\) and \((r_2, \psi_2)\) to \((\sqrt{2\alpha}, (5/3)\pi)\). See Figure 4. In our analysis we now interpret \(r\) and \(\psi\) in \((S2)\) as regular polar coordinates and proceed to determine the type of each critical point. It will turn out that \((r_1, \psi_1)\) is a stable node and \((r_2, \psi_2)\) a saddle point for all \((\mu, \alpha)\) in region \(I^+\). On \(L_1^+\) these two points coincide and form a saddle node.

The eigenvalues of the linearization at \((r_i, \psi_i), i = 1, 2\), are given by

\[
\lambda^N_i(\mu, \alpha) = -2r^2_i(\mu, \alpha), \quad \lambda^T_i(\mu, \alpha) = aq \cos \psi_i(\mu, \alpha).
\]

Associated eigenvectors with respect to the basis given by the \(r\)- and the \(\psi\)-directions are

\[
v^N_i(\mu, \alpha) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v^T_i(\mu, \alpha) = \begin{pmatrix} 2r_i \sin \psi_i(\mu, \alpha) \\ -2 + (q + 4) \cos \psi_i(\mu, \alpha) \end{pmatrix}.
\]

Figure 4. Linearizations of \((S2)\) at \((r_i, \psi_i), i = 1, 2\).
In region II+, $\lambda^\prime_\mu$ is different from $\lambda^\prime_\nu$ except for $(\mu, \alpha)$ belonging to the ray

$$L^+_* = \{(\mu, \alpha): \alpha = (1 - (2/(q + 4))^2)^{-1/2} \mu > 0\},$$

on which $v^\mu_1$ and $v^\nu_1$ are collinear. If we denote the angle $\psi_1$, defining $L^+_*$ by $\psi_1^* \in (\pi, (3/2)\pi)$, the critical point $(r_1, \psi_1^*)$ is a degenerate node. In Figure 4 we have sketched the curve $P_\alpha$ and the integral rays $\dot{\psi} = 0$, and have indicated the eigenspaces generated by $v^\nu_1$ for $(\mu, \alpha)$ on $L^+_*$ (Figure 4(a)), for $(\mu, \alpha)$ in II+, below $L^+_*$ (Figure 4(b)), and above $L^+_*$ (Figure 4(c)).

Because of the results in (2.1) and (2.5), the two branches of the unstable manifold of $(r_2, \psi_2)$ must join the critical point $(r_1, \psi_1)$ in the region $\{(r, \psi): 0 < r < r_0\}$.

![Figure 5](image.png)

**Figure 5.** Phase portraits of (S2) (a) on $L^+_1$, (b) in II+ with $\psi_1 > \psi_1^*$ and (c) in II+ with $\psi_1 < \psi_1^*$. 

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giving a homoclinic orbit in the situation of Figure 4(a) and two heteroclinic orbits for Figure 4(b), (c). In blown-up coordinates we thus have established the following: On $L^+_1$ and in $\Pi^+$ one has a unique invariant circle $C(\mu, \alpha)$ in $\{(r, \psi) : 0 < r < r_0\}$. The attractor of (S2) is the point $(r_*(\mu, \alpha), \psi_1(\mu, \alpha))$ on this circle. The circle $\{r = 0\}$ is repulsive. See Figure 5. For a general discussion of such a saddle node bifurcation, we refer to [2, pp. 360–362].

The problem is much easier to handle for negative values of $\alpha$. A similar analysis shows that the circle $\{r = 0\}$ is attractive on $L^-_1$, in $\Pi^-$, and on $L^-_2$. The attractor of (S2) is the critical point $(0, \psi_1(\mu, \alpha))$. See Figure 6. If we translate the results of 2.4 and 2.5 into the context of system (S) we have shown points (1.2.1), (1.2.2), (1.2.4) and (1.2.5).

2.6 The Naimark-Sacker bifurcations on $L^+_2$ and the regions $\Pi^\pm$. On $L^+_2$ the critical point $(r_1, \psi_1)$ undergoes a saddle node bifurcation with the critical point $(0, \psi_1)$ so that in region $\Pi^+$ and on the positive $\alpha$-axis, the unstable manifold of $(r_2, \psi_2)$ joins the critical point $(0, \psi_1)$ on the circle $\{r = 0\}$. The invariant circle $C(\mu, \alpha)$ now develops a cusp at $(0, \psi_1)$. See Figure 7.

We would like to remark that this saddle node bifurcation for the 2-dimensional system (S2) is, in fact, a Naimark-Sacker bifurcation for the 4-dimensional system

![Figure 6](image6.png)

**Figure 6.** Phase portraits of (S2) (a) on $L^-_1$, and (b) in $\Pi^-$ and on $L^-_2$.

![Figure 7](image7.png)

**Figure 7.** Phase portraits of (S2) (a) in $\Pi^+$, and (b) for $\mu = 0, \alpha > 0$. 
(S) or (S1). This follows immediately from an analysis of (S1). For \( \psi_1 \) close to \( (4/3)\pi \) and \( \psi_1 > (4/3)\pi \), the circle \( \{ (\psi, \phi): \psi = \psi_1 \} \) is repulsive; for \( \psi_1 \leq (4/3)\pi \), it is attractive. This leads to the contraction of the 2-torus \( \{ (r, \theta, \psi, \phi): r = r_1, \psi = \psi_1 \} \) to the circle \( \{ (\psi, \phi): \psi = \psi_1 \} \) when \( \psi_1 \) decreases through \( (4/3)\pi \).

For \( -\alpha > (2/\sqrt{3})\mu \geq 0 \) the critical points of (S2) are the stable node \( (0, \psi_1) \) and the saddle points \( (0, \psi_2) \) and \( (r_2, \psi_2) \). On \( L_2 \) the critical points \( (0, \psi_2) \) and \( (r_2, \psi_2) \) coincide and form a saddle node; in \( III^- \) and on the negative \( \alpha \)-axis \( r_2 \) is positive. As above, the unstable manifold of \( (r_2, \psi_2) \) joins the node \( (0, \psi_1) \). See Figure 8. Again, the translation into the 4-dimensional context of system (S) yields the claims of (1.2.3) and (1.2.6).

**Figure 8.** Phase portraits for (S2) (a) for \( \mu = 0, \alpha < 0 \), and (b) in \( III^- \).

### 3. Spectral properties of \( M^2(\mu, \alpha) \)

#### 3.1 Preliminaries.

We start by repeating the notions of the spectrum of an invariant manifold that are relevant for our analysis. For more details we refer to [6 and 7]. We take \( \psi \) and \( \phi \) as the coordinates on the invariant 2-torus \( M^2(\mu, \alpha) \) and denote the flow on \( M^2(\mu, \alpha) \) generated by

\[
\dot{\xi} = \begin{pmatrix}
q\mu + q\alpha \sin \psi \\
1 + a\mu + a\alpha \sin \psi
\end{pmatrix}, \quad \xi = \begin{pmatrix}
\psi \\
\phi
\end{pmatrix} \in T^2,
\]

by \( \sigma(\xi, \mu, \alpha, t) = (\Psi(\xi, \mu, \alpha, t), \alpha) \), \( \sigma(\xi, \mu, \alpha, 0) = \xi \). We choose \( \bar{\mu} > 0 \) and \( \bar{\alpha} > 0 \) so that \( \phi \) is positive for all \( (\xi, \mu, \alpha) \in T^2 \times [0, \bar{\mu}] \times [-\bar{\alpha}, \bar{\alpha}] \). Then we define the \( \lambda \)-shifted variational equations along \( \sigma \) for the normal and the tangential directions by

\[
(2)_\lambda \quad \dot{u} = (A(\Psi(\xi, \mu, \alpha, t), \alpha) - \lambda I)u, \quad A(\psi, \alpha) = \begin{pmatrix}
\alpha + 2\alpha \cos \psi & -\omega \\
\omega & \alpha + 2\alpha \cos \psi
\end{pmatrix},
\]

\[
(3)_\lambda \quad \dot{v} = (B(\Psi(\xi, \mu, \alpha, t), \alpha) - \lambda I)v, \quad B(\psi, \alpha) = \begin{pmatrix}
q\alpha \cos \psi & 0 \\
0 & a\alpha \cos \psi
\end{pmatrix}.
\]
respectively. For fixed $\xi \in T^2$ the normal portion $\Sigma^N(\xi, \mu, \alpha)$ and the tangential portion $\Sigma^T(\xi, \mu, \alpha)$ along the solution $\sigma(\xi, \mu, \alpha, t)$ of (1) are then

$$\Sigma^N(\xi, \mu, \alpha) = \{ \lambda \in \mathbb{R} : (2)_\lambda \text{ does not admit an exponential dichotomy} \},$$

$$\Sigma^T(\xi, \mu, \alpha) = \{ \lambda \in \mathbb{R} : (3)_\lambda \text{ does not admit an exponential dichotomy} \}.$$  

The normal portion $\Sigma^N(\mu, \alpha)$ and the tangential portion $\Sigma^T(\mu, \alpha)$ of the spectrum of the 2-torus $M^2(\mu, \alpha)$ are then the unions

$$\Sigma^N(\mu, \alpha) = \bigcup_{\xi \in \mathbb{R}^2} \Sigma^N(\xi, \mu, \alpha), \quad \Sigma^T(\mu, \alpha) = \bigcup_{\xi \in \mathbb{R}^2} \Sigma^T(\xi, \mu, \alpha).$$

3.2 The normal spectrum $\Sigma^N(\mu, \alpha)$. For $0 \leq |\alpha| < \mu$ the normal spectrum $\Sigma^N(\mu, \alpha)$ is equal to $\{ \alpha \}$ and the associated spectral subbundle is 2-dimensional. This follows easily from $(2)_\lambda$. We now take the case $|\alpha| > \mu$. In (2.2) we have determined two invariant circles $\Omega_i = \Omega_i(\mu, \alpha) = \{ \psi_i(\mu, \alpha) \} \times T^1$, $i = 1, 2$, for (1). Along the solutions $\sigma(\psi_i, \phi, \mu, \alpha, t)$ equation $(2)_\lambda$ is given by

$$\dot{u} = \begin{pmatrix} \alpha(1 + 2c) & -\omega \\ \omega & \alpha(1 + 2c) \end{pmatrix} u,$$

where we have taken $c$ to be $c(\mu, \alpha) = \cos \psi_2(\mu, \alpha) = \sqrt{1 - (\mu/\alpha)^2}$. The upper (lower) sign refers to $\psi_1$ ($\psi_2$, respectively). For any solution of (1) with initial value $\xi = (\psi, \phi)$, $\psi_1 \neq \psi_2$, the positive limit set is $\Omega_1$, the negative limit set $\Omega_2$. Because of Lemma 1 of [6] we can conclude that the points $\alpha(1 \pm 2c)$ belong to $\Sigma^N(\xi, \mu, \alpha)$. Since the dimensions of the stable subbundle over $\Omega_1$ and the unstable over $\Omega_2$ are both 2, Theorem 3.2 of [5] implies that the interval with $\alpha(1 \pm 2c)$ as endpoints belongs to $\Sigma^N(\xi, \mu, \alpha)$. For any $\lambda$ outside this interval $(2)_\lambda$ admits an exponential dichotomy. In summarizing we can say that $\Sigma^N(\mu, \alpha)$ is given by $\{ \alpha \}$ for $0 \leq |\alpha| < \mu$ and by

$$\Sigma^N(\mu, \alpha) = \begin{cases} \left[ \alpha(1 - 2c(\mu, \alpha)), \alpha(1 + 2c(\mu, \alpha)) \right] & \text{for } \alpha > \mu \geq 0, \\
\left[ \alpha(1 + 2c(\mu, \alpha)), \alpha(1 - 2c(\mu, \alpha)) \right] & \text{for } -\alpha > \mu \geq 0, \end{cases}$$

and that the associated spectral subbundle is 2-dimensional (cf. [5, 6]).

3.3 The tangential spectrum $\Sigma^T(\mu, \alpha)$. For $0 \leq |\alpha| < \mu$ the 2-torus $M^2(\mu, \alpha)$ is minimal and the only bounded solution of $(3)_\lambda$, $\lambda \neq 0$, is the trivial solution. For $\lambda = 0$ all solutions of $(3)_\lambda$ are bounded. Therefore the tangential spectrum $\Sigma^T(\mu, \alpha)$ is equal to $\{ 0 \}$. For $|\alpha| > \mu \geq 0$ there are two invariant circles $\Omega_i(\mu, \alpha)$ on $M^2(\mu, \alpha)$. Again, for any solution of (1) with initial value $\xi = (\psi, \phi)$, $\psi_1 \neq \psi \neq \psi_2$, the positive limit set is $\Omega_1$, the negative $\Omega_2$. The same arguments as in (3.2) lead to

$$\Sigma^T(\mu, \alpha) = \begin{cases} -q|\alpha|c(\mu, \alpha), q|\alpha|c(\mu, \alpha) \end{cases}$$

for $|\alpha| > \mu \geq 0$. 

3.4 Consequences. (1) From (3.2) it follows that for $\mu = 0$, $\alpha \neq 0$ the normal spectrum is an interval containing 0 in its interior. For any fixed positive $\mu$, $\Sigma^N(\mu, \alpha)$ is in $\mathbb{R}^-$ for $-(2/\sqrt{3})\mu < \alpha < 0$, reduces to $\{ 0 \}$ for $\alpha = 0$ and is in $\mathbb{R}^+$ for $0 < \alpha < (2/\sqrt{3})\mu$. Thus the underlying invariant 2-torus $M^2(\mu, \alpha)$ is normally attractive for $-(2/\sqrt{3})\mu < \alpha < 0$ and normally repulsive for $0 < \alpha < (2/\sqrt{3})\mu$. As
α increases (decreases) through \((2/\sqrt{3})\mu\) \((-2/\sqrt{3})\mu\) the left (right) endpoint of \(\Sigma^N(\mu, \alpha)\) crosses through 0 again. This proves (1.3.1) and (1.3.2).

(2) The spectra \(\Sigma^N(\mu, \alpha)\) and \(\Sigma^T(\mu, \alpha)\) are disjoint for

\[0 < |\alpha| < \left(1 - (1/(q + 2))^2\right)^{-1/2} \mu\]

and \(M^2(\mu, \alpha)\) is normally hyperbolic for these \((\mu, \alpha)\). Note that the set of \((\mu, \alpha)\) defined by the above inequality contains the regions \(I^+\) and \(I^-\) and parts of \(II^+\) and \(II^-\).

(3) It is of interest to note that for \(\alpha > (2/\sqrt{3})\mu\), the left endpoint of \(\Sigma^T(\mu, \alpha)\) is always smaller than the one of \(\Sigma^N(\mu, \alpha)\) so that the tangential flow at the attractor on \(M^2(\mu, \alpha)\) is faster than the normal flow. That this leads to a cusp of the pinched 3-torus has been shown in (2.6). For \(\alpha < -(2/\sqrt{3})\mu\) the same holds for \(q \geq 4\). For \(q = 2\) or \(q = 3\) (and \(\mu > 0\)) the pinched 3-torus is flat where it is pinched.

REFERENCES


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