A GROUP-THEORETIC CHARACTERIZATION OF M-GROUPS

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Abstract. Groups having the property that all their complex irreducible characters are monomial are characterized in terms of the embedding of cyclic sections of the group.

Introduction. A character of a finite group $G$ is monomial if it is induced from a linear (degree-one) character of a subgroup of $G$. The group $G$ is an $M$-group if all of its complex irreducible characters (the set $\text{Irr}(G)$) are monomial.

Isaacs [5, and 4, p. 67] and Berger [1, p. 43] have asked for a purely group-theoretic characterization of $M$-groups. We will now describe such a characterization; proofs will be provided in §1.

If $M<H \leq G$ with $H/M$ cyclic, we will say that $(H, M)$ is a pair. For $g \in G$ and $H \leq G$ we define $F_H(g)$ to be the set of commutators $[g, H \cap H^g]$. We note that $F_H(g) \subseteq H$: indeed if $h \in H \cap H^g$, then $h = gkg^{-1}$ for some $k \in H$. Then $[g, h] = g^{-1}h^{-1}gh = k^{-1}h \in H$. If $(H, M)$ is a pair, we will say that it is a good pair in $G$, if $F_H(g) \subseteq M$ for all $g \in G - H$.

If $(H, M)$ and $(K, L)$ are good pairs, we will say they are related in $G$ if there is $g \in G$ such that $H^g \cap L = K \cap M^g$. Let $S_G$ be the equivalence relation on good pairs in $G$ generated by the relation of being related. Let $m_G$ be the number of distinct classes of $S_G$.

We identify a relation on the elements of $G$. We say $x \sim y$ for $x, y \in G$ provided the two cyclic groups $\langle x \rangle$ and $\langle y \rangle$ are conjugate in $G$. Clearly $\sim$ is an equivalence relation. (The equivalence classes of $\sim$ are sometimes called the rational conjugacy classes of $G$.) Let $n_G$ be the number of $\sim$ equivalence classes.

Theorem. We have $m_G \leq n_G$ with equality if and only if $G$ is an $M$-group.

The Theorem is the promised characterization.

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1. Proofs. Let $J$ and $L$ be subgroups of a group $G$. A set of representatives $T$ for the double cosets of $J$ and $L$ in $G$ will be called a $J, L$ transversal in $G$.

For a character $\theta$ of $J$ and $x \in G$ we define a character $\theta^x$ of $J^x$ by the formula

$$\theta^x(g) = \theta(xgx^{-1}) \quad \text{for} \quad g \in J^x.$$
1.0 Theorem (Mackey). Let $J, L \subseteq G$. Let $T$ be a $J, L$ transversal in $G$. Let $\theta$ and $\varphi$ be characters of $J$ and $L$, respectively. Then

$$[\theta^G, \varphi^G] = \sum_{g \in T} [\theta^g]_{J^g \cap L^g, \varphi^g_{J^g \cap L^g}}. \quad \Box$$

For any pair $(H, M)$, there is a linear $\lambda \in \text{Irr}(H)$ with $M$ equal to the kernel of any representation affording $\lambda$ (we write $M = \text{ker}(\lambda)$). We will say that $\lambda$ proceeds from $(H, M)$.

1.1 Proposition. Let $(H, M)$ be a pair with $H \subseteq G$. Let $\lambda$ proceed from $(H, M)$. Then $(H, M)$ is a good pair in $G$ if and only if the induced character $\lambda^G$ is irreducible.

Proof. Let $\lambda$ proceed from $(H, M)$.

Claim. If $x \in G$ then $[(\lambda^x)_{H^x \cap H}, \lambda_{H^x \cap H}] = 1$ if and only if $F_H(x^{-1}) \subseteq M$.

Proof. Put $K = H^x \cap H$. Then $\lambda^x$ and $(\lambda^x)_K$ are linear characters of $K$. Hence $[(\lambda^x)_K, \lambda_K] = 1$ if and only if $(\lambda^x)_K = \lambda_K$.

Let $g \in K$ and suppose $(\lambda^x)_K = \lambda_K$. Then $\lambda^x(g) = \lambda(g)$, so then $\lambda(xgx^{-1}) = \lambda(g)$. Since $\lambda$ is linear, this proves that $\lambda(xgx^{-1}g^{-1}) = 1$, and so $[x^{-1}, g^{-1}] \in \text{ker}(\lambda) = M$. Thus $F_H(x^{-1}) = [x^{-1}, K] \subseteq M$. Conversely, if $F_H(x^{-1}) \subseteq M$, then $\lambda(xgx^{-1}) = \lambda(g)$ for all $g \in K$. Then $(\lambda^x)_K = \lambda_K$, as needed. \(\Box\)

Now $\lambda^G \in \text{Irr}(G)$ iff $[\lambda^G, \lambda^G] = 1$. Let $\lambda^G$ be irreducible and choose $x \in G - H$. Then there is an $H, H$ transversal $T$ in $G$ with $1, x \in T$. By Theorem 1.0

$$[\lambda^G, \lambda^G] \geq [\lambda_H, \lambda_H] + [(\lambda^x)_{H^x \cap H}, \lambda_{H^x \cap H}].$$

So then $[(\lambda^x)_{H^x \cap H}, \lambda_{H^x \cap H}] = 0$. By the Claim, $F_H(x) \not\subseteq M$. This proves one direction of Proposition 1.1.

Suppose for all $x \in G - H$ that $F_H(x) \not\subseteq M$. By the Claim, $[(\lambda^x)_{H^x \cap H}, \lambda_{H^x \cap H}] = 0$ for all $x \in G - H$. Then using Theorem 1.0 we see that $[\lambda^G, \lambda^G] = [\lambda_H, \lambda_H] = 1$. This completes the proof of Proposition 1.1. \(\Box\)

1.2 Proposition. If $(H, M)$ and $(K, L)$ are good pairs, then they are related if and only if there are characters $\lambda$ and $\mu$ proceeding from $(H, M)$ and $(K, L)$, respectively, such that $\lambda^G = \mu^G$.

Proof. Assume $\lambda$ and $\mu$ proceed from the good pairs $(H, M)$ and $(K, L)$, respectively, and suppose that $\lambda^G = \mu^G$.

Let $T$ be an $H, K$ transversal in $G$. By Theorem 1.0, since $[\lambda^G, \mu^G] \neq 0$, we have

$$[\lambda^x_{H^x \cap K}, \mu^x_{H^x \cap K}] \neq 0 \quad \text{for some } x \in T.$$

Now $(\lambda^x)_{H^x \cap K}$ and $\mu^x_{H^x \cap K}$ are linear and we conclude that $(\lambda^x)_{H^x \cap K} = \mu^x_{H^x \cap K}$. In particular, their kernels are the same, that is

$$M^x \cap H^x \cap K = L \cap H^x \cap K.$$

This is clearly $M^x \cap K = L \cap H^x$, Hence $(H, M)$ and $(K, L)$ are related.

Conversely, assume $H^x \cap L = K \cap M^x$ for some $x \in G$. Then $L \cap H^x \cap K = H^x \cap K \cap M^x$; call this group $N$. Now $(H^x \cap K)/N$ is isomorphic to a subgroup of
characterization of $M$-groups

$K/L$ which is cyclic. Thus there is a faithful linear $\nu \in \text{Irr}((H^x \cap K)/N)$. Since $N = (H^x \cap K) \cap L$, $\nu$ extends to $\mu \in \text{Irr}(K)$ with $L = \text{ker}(\mu)$, and since $N = (H^x \cap K) \cap M^x$, $\nu$ extends to $\lambda^x \in \text{Irr}(H^x)$, where $\lambda \in \text{Irr}(H)$ and $M = \text{ker}(\lambda)$.

Including $x$ in an $H, K$ transversal in $G$, Theorem 1.0. shows that

$$[\lambda^G, \mu^G] \geq [(\lambda^x)^{H^x \cap K}, \mu^{H^x \cap K}] = [\nu, \nu] = 1.$$ 

Because $(H, M)$ and $(K, L)$ are good pairs, $\lambda^G$ and $\mu^G$ are irreducible. Thus $\lambda^G = \mu^G$ as needed. □

We remark that Proposition 1.2 shows that being related is actually an equivalence relation on the set of good pairs, and so the equivalence classes of $S_G$ are precisely the classes of related good pairs. It might be interesting to find a purely group-theoretic proof that being related is an equivalence relation.

The proof of the Theorem is close at hand. We say $\chi, \psi \in \text{Irr}(G)$ are Galois conjugate if there is $s \in \text{Aut}(C)$ such that $\chi^s = \psi$. If $s(\chi)$ is the Schur index of $\chi$ over the rationals (see [4, Chapter 10]), then $s(\chi)$ times the sum $\text{sp}(\chi)$ of the distinct Galois conjugates of $\chi$ in $\text{Irr}(G)$ is the character afforded by an irreducible, rational representation of $G$. By [4, Theorem 9.21], all irreducible, rationally-afforded characters of $G$ arise as $s(\chi)\text{sp}(\chi)$ for $\chi \in \text{Irr}(G)$. By the Berman-Witt Theorem [2, 42.9], the number $n_G$ defined in the Introduction is the same as the number of distinct, irreducible, rationally-afforded characters of $G$, and thus $n_G$ is the number of Galois conjugacy classes of $\text{Irr}(G)$.

**Proof of Theorem.** By the preceding discussion, it suffices to show that there is a one-to-one correspondence between the set of Galois conjugacy classes of monomial elements of $\text{Irr}(G)$ and classes of related good pairs.

Let $(H_i, M_i), 1 \leq i \leq m_G$, be a set of representatives of the classes of related good pairs in $G$. For each $i$, let $\lambda_i$ proceed from $(H_i, M_i)$; then $\lambda^G_i \in \text{Irr}(G)$ by Proposition 1.1. To complete the proof, we will show that, given monomial $\chi \in \text{Irr}(G)$, there is a unique $i$ for which $\chi$ is Galois conjugate to $\lambda^G_i$.

Indeed, suppose $\chi = \mu^G$ where $\mu \in \text{Irr}(H), H \subseteq G$, and $\mu(1) = 1$. Put $M = \text{ker}(\mu)$; then by Proposition 1.1, $(H, M)$ is a good pair from which $\mu$ proceeds. Now $(H, M)$ is related to some $(H_i, M_i)$ and Proposition 1.2 grants $\mu'$ proceeding from $(H, M)$ and $\lambda'$ proceeding from $(H_i, M_i)$ with

\[ (\mu')^G = (\lambda')^G. \]

The characters $\mu'$ and $\mu$ faithfully represent the same cyclic group. By the irreducibility of the cyclotomic polynomials there is $\sigma \in \text{Aut}(C)$ such that $\mu^\sigma = \mu'$. Similarly there is $\tau \in \text{Aut}(C)$ with $(\lambda')^\tau = \lambda_i$. Compute

$$\chi^{\sigma\tau} = ((\mu^\sigma)^G)^\tau = ((\mu')^G)^\tau = ((\lambda')^G)^\tau \quad \text{using } (\ast)$$

$$= ((\lambda')^G)^G = \lambda^G_i.$$ 

Thus $\chi$ is Galois conjugate to $\lambda^G_i$.
As for the uniqueness of \( i \), if \( \chi \) is also conjugate to \( \lambda_j^G \), then there is \( \sigma \in \text{Aut}(C) \) with \( (\lambda_j^G)^\sigma = \lambda_j^G \). Thus \( (\lambda_i^G)^G = \lambda_j^G \). Observe that \( \lambda_i^G \) proceeds from \( (H_i, M_i) \), and then Proposition 1.2 allows us to conclude that \( (H_i, M_i) \) and \( (H_j, M_j) \) are related. This forces that \( i = j \). The proof is complete. \( \square \)

References


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