TRANSLATION-INVARIANT LINEAR FORMS
ON $L_p(G)$

JOSEPH ROSENBLATT

Abstract. Let $G$ be a compact group such that the identity representation of $G$ is not contained in the regular representation on $L^2(G, \lambda_G)$ of $G$ with the discrete topology. Then any left translation invariant linear form on $L_p(G)$, $1 < p < \infty$, is continuous and must be a constant times the Haar integral. This shows that many classical matrix groups $G$ admit only continuous left translation invariant linear forms on $L_p(G)$, $1 < p < \infty$.

Let $G$ be a compact Hausdorff group and let $\lambda_G$ be the normalized Haar measure on $G$. A linear form on $L_p(G)$ is a functional $\varphi$ on $L_p(G)$ which is linear. Given $g \in G$ and $f \in L_p(G)$, define $gf(x) = f(g^{-1}x)$ for a.e. $x \in G$. We say that the linear form $\varphi$ is invariant if $\varphi(gf) = \varphi(f)$ for all $g \in G$. Under what conditions on $G$ is a linear form on $L_p(G)$ automatically continuous? This problem has received considerable attention for a variety of groups and function spaces. If $G$ is a connected compact abelian group, Meisters and Schmidt [6] showed that any invariant linear form on $L_2(G)$ is continuous. This was recently extended by Bourgain [1] to $L_p(T)$, $1 < p < \infty$. On the other hand, Meisters [5] has shown that some totally-disconnected compact groups have discontinuous invariant linear forms on $L_p(G)$. The examples given here are for quite different groups than have been studied previously in this context.

We say $G$ has the mean-zero weak containment property if for all $g_1, \ldots, g_n \in G$, and $\varepsilon > 0$, there exists $f \in L^0_2(G) = \{ f \in L_2(G) : \int f d\lambda_G = 0 \}$ such that $\| f \|_2 = 1$ and $\| g f - f \|_2 < \varepsilon$ for $i = 1, \ldots, n$. That is, the identity representation of $G$ is contained in the regular representation on $L^2(G, \lambda_G)$ of $G$ with the discrete topology. If $G$ is amenable as a discrete group, then $G$ has this property. On the other hand, if $G$ contains a dense discrete subgroup with Kazhdan’s property $T$, then $G$ does not have the mean-zero weak containment property. Recently, in solving the Banach-Ruziewicz problem for $S^2$ and $S^3$, V. G. Drinfeld [3] has shown that $\text{SO}(3)$ and $\text{SO}(4)$ do not have the mean-zero weak containment property. This, together with Margulis [4], shows that $\text{SO}(n)$, $n \geq 3$, does not have the mean-zero weak containment property. Moreover, it follows from [3, 4] that any compact simple Lie group does not have the mean-zero weak containment property. See [2, 7] for a
discussion of this property and its relationship to the uniqueness of invariant means on \( L_\infty(G) \).

**Lemma.** Suppose \( G \) does not have the mean-zero weak containment property. Then there exists \( g_1, \ldots, g_n \in G \) such that for some \( \delta_p < 1 \) and any \( f \in L_p^0(G) \), \( 1 < p < \infty \), we have

\[
\left\| \frac{1}{n+1} \left( f + \sum_{i=1}^n \delta_{g_i} f \right) \right\|_p \leq \delta_p \|f\|_p.
\]

**Proof.** Let \( \mu = (1/(n+1))(\delta_e + \sum_{i=1}^n \delta_{g_i}) \) where \( e \) is the identity in \( G \) and \( \delta_g \), \( g \in G \), denotes the Dirac mass measure at \( g \). Then \( \mu \) acts by convolution on \( L_p^0(G) \) with \( \|\mu\|_{L_p^0} \leq 1 \). Suppose \( \|\mu\|_{L_p^0} = 1 \). Then there exists a sequence \( (f_m) \subset L_0^0(G) \) such that \( \|f_m\|_2 = 1 \) for all \( m \geq 1 \), and \( \lim_{m \to \infty} \|\mu \ast f_m\|_2 = 1 \). It is easy to show, as in [2, Theorem 1.1], that this forces

\[
\lim_{m \to \infty} \left\| g, f_m - f_m \right\|_2 = 0 \quad \text{for each } i = 1, \ldots, n.
\]

So if \( G \) does not have the mean-zero weak containment property, there exists \( \mu \) as above with \( \|\mu\|_{L_p^0} < 1 \). An interpolation argument as in [8] shows \( \|\mu\|_{L_p^0} < 1 \) for all \( p \), \( 1 < p < \infty \). \( \square \)

**Proposition.** Suppose \( G \) does not have the mean-zero weak containment property. Then there exists \( g_1, \ldots, g_n \in G \) such that for every \( f \in L_p(G) \), \( 1 < p < \infty \), there exists \( h \in L_0^0(G) \) such that

\[
f = \left( \int d\lambda_G \right) 1_G + \sum_{i=1}^n (h - g_i h).
\]

**Proof.** Let \( \mu \) be as in the proof of the lemma. Denote by \( \mu^n, n \geq 1 \), the \( n \)-th-convolution power of \( \mu \), and let \( \mu^0 = \delta_e \). Let \( f_0 \in L^0_p(G) \). Since \( \|\mu\|_{L_p^0} < 1 \), the series \( k = \sum_{n=1}^\infty \mu^n \ast f_0 \) converges in \( L_p^0(G) \). Also, \( k - \mu \ast k = \mu^0 \ast \mu \ast f_0 = f_0 \). If \( h = k/(n+1) \), then

\[
f_0 = f_0 = f - (f \ast d\lambda_G) 1_G = f_0.
\]

If \( f \in L_p(G) \), let \( f_0 = f - (f \ast d\lambda_G) 1_G \) to get the representation of the proposition. \( \square \)

**Theorem.** Suppose \( G \) does not have the mean-zero weak containment property. Then there exists \( g_1, \ldots, g_n \in G \) such that any linear form on \( L_p(G) \), \( 1 < p < \infty \), invariant under \( g_1, \ldots, g_n \) must be continuous and therefore a scalar times the Haar integral.

**Proof.** Let \( g_1, \ldots, g_n \in G \) as in the proposition. Let \( \phi \) be a linear form invariant under \( g_1, \ldots, g_n \). Then, using the representation of the proposition, \( \phi(f) = \phi(1_G) \int \ast d\lambda_G \). \( \square \)

**Remark 1.** Suppose \( G \) is abelian and \( H \) is a countable subgroup of \( G \). Then [8, Theorem 14] shows that

\[
S = \text{span} \{ g, f - f : g \in H, f \in L_p(G) \}, \quad 1 < p < \infty.
\]
is not closed. Hence, there exists discontinuous $H$ invariant linear forms on $L_p(G)$, $1 < p < \infty$. This shows that the result of Meisters and Schmidt [6] requires the full invariance under $G$, and contrasts the result above with previous results in that the invariance hypothesis is weakened.

**Remark 2.** A theorem similar to the above is true for an ergodic group action on a probability space where the action does not have the mean-zero weak containment property. See Proposition 11 and the remarks after Proposition 13 in [8]. For example, let $\varphi$ be a linear form on $L_p(T^2)$, $1 < p < \infty$, where $T$ denotes the circle group. Let $\tau_1, \tau_2 : T^2 \to T^2$ be defined by $\tau_1(z_1, z_2) = (z_2, z_1)$, $\tau_2(z_1, z_2) = (z_1z_2, z_2)$ for all $z_1, z_2 \in T$. If $\varphi$ is a linear form on $L_p(T^2)$, $1 < p < \infty$, such that $\varphi(f \circ \tau_i) = \varphi(f)$ for all $f \in L_p(T^2)$ and $i = 1, 2$, then $\varphi$ is continuous and a scalar times the Haar integral.

**References**


Department of Mathematics, Ohio State University, Columbus, Ohio 43210