TRANSLATION-INVARIANT LINEAR FORMS
ON $L^p(G)$

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Abstract. Let $G$ be a compact group such that the identity representation of $G$ is not contained in the regular representation on $L^2(G, \lambda_G)$ of $G$ with the discrete topology. Then any left translation invariant linear form on $L^p(G)$, $1 < p < \infty$, is continuous and must be a constant times the Haar integral. This shows that many classical matrix groups $G$ admit only continuous left translation invariant linear forms on $L^p(G)$, $1 < p < \infty$.

Let $G$ be a compact Hausdorff group and let $\lambda_G$ be the normalized Haar measure on $G$. A linear form on $L^p(G)$ is a functional $\varphi$ on $L^p(G)$ which is linear. Given $g \in G$ and $f \in L^p(G)$, define $gf(x) = f(g^{-1}x)$ for a.e. $x \in G$. We say that the linear form $\varphi$ is invariant if $\varphi(gf) = \varphi(f)$ for all $g \in G$. Under what conditions on $G$ is a linear form on $L^p(G)$ automatically continuous? This problem has received considerable attention for a variety of groups and function spaces. If $G$ is a connected compact abelian group, Meisters and Schmidt [6] showed that any invariant linear form on $L^2(G)$ is continuous. This was recently extended by Bourgain [1] to $L^p(T)$, $1 < p < \infty$. On the other hand, Meisters [5] has shown that some totally-disconnected compact groups have discontinuous invariant linear forms on $L^p(G)$. The examples given here are for quite different groups than have been studied previously in this context.

We say $G$ has the mean-zero weak containment property if for all $g_1, \ldots, g_n \in G$, and $\varepsilon > 0$, there exists $f \in L^0_2(G) = \{ f \in L^2(G): \int f \, d\lambda_G = 0 \}$ such that $\|f\|_2 = 1$ and $\|g_if - f\|_2 < \varepsilon$ for $i = 1, \ldots, n$. That is, the identity representation of $G$ is contained in the regular representation on $L^2_2(G, \lambda_G)$ of $G$ with the discrete topology. If $G$ is amenable as a discrete group, then $G$ has this property. On the other hand, if $G$ contains a dense discrete subgroup with Kazhdan’s property $T$, then $G$ does not have the mean-zero weak containment property. Recently, in solving the Banach-Ruziwieicz problem for $S^2$ and $S^3$, V. G. Drinfeld [3] has shown that $SO(3)$ and $SO(4)$ do not have the mean-zero weak containment property. This, together with Margulis [4], shows that $SO(n)$, $n \geq 3$, does not have the mean-zero weak containment property. Moreover, it follows from [3, 4] that any compact simple Lie group does not have the mean-zero weak containment property. See [2, 7] for a
discussion of this property and its relationship to the uniqueness of invariant means on $L_\infty(G)$.

**Lemma.** Suppose $G$ does not have the mean-zero weak containment property. Then there exists $g_1, \ldots, g_n \in G$ such that for some $\delta_p < 1$ and any $f \in L^0_p(G)$, $1 < p < \infty$, we have

$$\left\lVert \frac{1}{n+1} \left( f + \sum_{i=1}^n \delta g_i f \right) \right\rVert_p \leq \delta_p \lVert f \rVert_p.$$

**Proof.** Let $\mu = (1/(n+1))(\delta_e + \sum_{i=1}^n \delta g_i)$ where $e$ is the identity in $G$ and $\delta g_i$, $g \in G$, denotes the Dirac mass measure at $g$. Then $\mu$ acts by convolution on $L^0_p(G)$ with $\lVert \mu \rVert_{L^0_p} < 1$. Suppose $\lVert \mu \rVert_{L^0_2} = 1$. Then there exists a sequence $(f_m) \in L^0_0(G)$ such that $\lVert f_m \rVert_2 = 1$ for all $m \geq 1$, and $\lim_{m \to \infty} \lVert \mu \ast f_m \rVert_2 = 1$. It is easy to show, as in [2, Theorem 1.1], that this forces

$$\lim_{m \to \infty} \lVert g_i f_m - f_m \rVert_2 = 0 \quad \text{for each } i = 1, \ldots, n.$$  

So if $G$ does not have the mean-zero weak containment property, there exists $\mu$ as above with $\lVert \mu \rVert_{L^0_2} < 1$. An interpolation argument as in [8] shows $\lVert \mu \rVert_{L^p_2} < 1$ for all $p$, $1 < p < \infty$. \qed

**Proposition.** Suppose $G$ does not have the mean-zero weak containment property. Then there exists $g_1, \ldots, g_n \in G$ such that for every $f \in L_p(G)$, $1 < p < \infty$, there exists $h \in L^0_p(G)$ such that

$$f = \left( \int f d\lambda_G \right) 1_G + \sum_{i=1}^n (h - g_i h).$$

**Proof.** Let $\mu$ be as in the proof of the lemma. Denote by $\mu^n$, $n \geq 1$, the $n$th-convolution power of $\mu$, and let $\mu^0 = \delta_e$. Let $f_0 \in L^0_p(G)$. Since $\lVert \mu \rVert_{L^0_p} < 1$, the series $k = \sum_{n=1}^\infty \mu^n \ast f_0$ converges in $L^0_p(G)$. Also, $k - \mu \ast k = \mu^0 \ast f_0 = f_0$. If $h = k/(n+1)$, then

$$f_0 = \frac{nk}{n+1} - \frac{1}{n+1} \sum_{i=1}^n g_i k = nh - \sum_{i=1}^n g_i h = \sum_{i=1}^n (h - g_i h).$$

If $f \in L_p(G)$, let $f_0 = f - (\int f d\lambda_G) 1_G$ to get the representation of the proposition. \qed

**Theorem.** Suppose $G$ does not have the mean-zero weak containment property. Then there exists $g_1, \ldots, g_n \in G$ such that any linear form on $L_p(G)$, $1 < p < \infty$, invariant under $g_1, \ldots, g_n$ must be continuous and therefore a scalar times the Haar integral.

**Proof.** Let $g_1, \ldots, g_n \in G$ as in the proposition. Let $\phi$ be a linear form invariant under $g_1, \ldots, g_n$. Then, using the representation of the proposition, $\phi(f) = \phi(1_G) \int f d\lambda_G$. \qed

**Remark 1.** Suppose $G$ is abelian and $H$ is a countable subgroup of $G$. Then [8, Theorem 14] shows that

$$S = \text{span}\left\{ g f - f : g \in H, f \in L_p(G) \right\}, \quad 1 < p < \infty.$$
is not closed. Hence, there exists discontinuous $H$ invariant linear forms on $L_p(G)$, $1 < p < \infty$. This shows that the result of Meisters and Schmidt [6] requires the full invariance under $G$, and contrasts the result above with previous results in that the invariance hypothesis is weakened.

**Remark 2.** A theorem similar to the above is true for an ergodic group action on a probability space where the action does not have the mean-zero weak containment property. See Proposition 11 and the remarks after Proposition 13 in [8]. For example, let $\varphi$ be a linear form on $L_p(T^2)$, $1 < p < \infty$, where $T$ denotes the circle group. Let $\tau_1, \tau_2: T^2 \to T^2$ be defined by $\tau_1(z_1, z_2) = (z_2, z_1)$, $\tau_2(z_1, z_2) = (z_1z_2, z_2)$ for all $z_1, z_2 \in T$. If $\varphi$ is a linear form on $L_p(T^2)$, $1 < p < \infty$, such that $\varphi(f \circ \tau_i) = \varphi(f)$ for all $f \in L_p(T^2)$ and $i = 1, 2$, then $\varphi$ is continuous and a scalar times the Haar integral.

**References**


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