ON THE MULTIPLICITIES OF THE POWERS OF A BANACH SPACE OPERATOR

DOMINGO A. HERRERO

ABSTRACT. The multiplicities of the powers of a bounded linear operator $T$, acting on a complex separable infinite-dimensional Banach space $X$, satisfy the inequalities

$$\mu(T^n) \leq h\mu(T^h) \leq \mu(T^h)$$

for all $h, n \geq 1$.

Nothing else can be said, in general, because simple examples show that for each sequence $\{\mu_n\}_{n=1}^\infty$, satisfying the inequalities (••), there exists $T$ acting on $X$ such that $\mu(T^n) = \mu_n$ for all $n \geq 1$.

Let $\mathcal{L}(X)$ denote the algebra of all (bounded linear) operators acting on a complex separable infinite-dimensional Banach space $X$. The multiplicity of $T$ in $\mathcal{L}(X)$ is the cardinal number defined by

$$\mu(T) = \min_{\Gamma \subset X} \left\{ \mu(T^n) : y \in \Gamma, n = 0, 1, 2, \ldots \right\}.$$

where $\vee\mathcal{R}$ denotes the closed linear span of the vectors in $\mathcal{R}$. It is immediate from the definition that

$$\mu(T^n) \leq \mu(T^h) \leq h\mu(T^h) \quad \text{for all} \ h, n \geq 1.$$

Thus, in particular, if $T^n$ is cyclic (i.e., $\mu(T^n) = 1$) for some $n \geq 1$, then $T^m$ is also cyclic for all $m|n$ (where $m|n$ indicates that $m$ divides $n$).

This is the only possible general result relating the multiplicities of the different powers of a given operator. Indeed, we have the following result.

**Theorem 1.** Let $X$ be a complex separable infinite-dimensional Banach space. Given a sequence $M = \{\mu_n\}_{n=1}^\infty$ of natural numbers, satisfying the inequalities

$$\mu_n \leq h\mu_n \leq \mu_n$$

for all $h, n \geq 1$,

there exists a nuclear operator $T(M)$ in $\mathcal{L}(X)$ such that

$$\mu(T(M)^n) = \mu_n \quad \text{for all} \ n = 1, 2, \ldots.$$

For the Hilbert space case, we have

**Theorem 2.** Let $H$ be a complex separable infinite-dimensional Hilbert space. Given a sequence $M$ satisfying the conditions of Theorem 1, there exists a normal operator $N(M)$ in $\mathcal{L}(H)$ such that

$$\mu(N(M)^n) = \mu_n \quad \text{for all} \ n = 1, 2, \ldots, \quad \text{and} \quad \sigma(N(M)) = \{\lambda : |\lambda| \leq 1\},$$

where $\sigma(N(M))$ denotes the spectrum of $N(M)$.

Received by the editors February 27, 1984 and, in revised form, July 6, 1984.

1980 Mathematics Subject Classification. Primary 47A99, 47B10.

Key words and phrases. Multiplicity of an operator, powers.

1 This research was partially supported by a grant from the National Science Foundation

©1985 American Mathematical Society

0002-9939/85 $1.00 + $.25 per page

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

239
Let \( \{ e_i \}_{i=1}^{\mu_n} \) be the canonical orthonormal basis of \( \mathbb{C}^{\mu_n} \) and let \( e(t) = \exp\{2\pi it\} \) (\( t \) a real number). If \( \alpha_0 + \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} = \mu_n \), then we write
\[
A = A_0, A_1, A_2, \ldots, A_{n-1} = \bigoplus_{j=0}^{\mu_n-1} e\left( \frac{j}{n} \right)
\]
to indicate the diagonal (and therefore normal) operator defined by
\[
Ae_i = \begin{cases} e_i, & 1 \leq i \leq \alpha_0, \\ e\left( \frac{j}{n} \right)e_i, & \alpha_0 + \alpha_1 + \cdots + \alpha_{j-1} < i \leq \alpha_0 + \alpha_1 + \cdots + \alpha_j \\ 0, & j = 1, 2, \ldots, n - 1. \end{cases}
\]
Observe that \( A^\mu \) is the identity operator, and therefore
\[
\mu(A^\mu) = \dim \mathbb{C}^{\mu_n} = \mu_n.
\]
It is clear that every operator satisfying this condition must also satisfy
\[
\mu(A^m) \leq \left\{ \text{smallest integer greater than or equal to } m\mu_n/n \right\} = \left\lfloor (m\mu_n + n - 1)/n \right\rfloor + 1
\]
for each \( m | n \), where \( \lfloor t \rfloor \) denotes the integral part of the real number \( t \). (To see this, use (1).)

Observe that, for each \( k \geq 1 \),
\[
\mu(A^k) = \mu\left( \bigoplus_{j=0}^{\sigma_n-1} e\left( \frac{j}{n} \right) \right) = \max_{0 \leq t < n} \sum_{j=0}^{\sigma_n-1} \alpha_j : kj \equiv t \pmod{n}
\]
(with the convention that \( \sum_{j=0}^{\sigma_n-1} \alpha_j : kj \equiv t \pmod{n} = 0 \) if \( kj \not\equiv t \pmod{n} \) for all \( j = 0, 1, 2, \ldots, n - 1 \).

The key result is Lemma 3 below, which says that, for a clever choice of \( \alpha_0, \alpha_1, \ldots, \alpha_{n-1} \), \( \mu(A^k) \) does not exceed \( \mu(B^k) \) for any operator \( B \) such that \( \mu(B^\mu) = \mu_n \) (for all \( k = 1, 2, \ldots \)).

**Lemma 3.** Let \( n, \mu_n \geq 1 \) and let
\[
s_n = \left\lfloor (\mu_n - 1)/n \right\rfloor + 1 \quad \text{and} \quad a_n = \mu_n - n(s_n - 1).
\]
If
\[
A_n = \left( \bigoplus_{i=0}^{a_n-1} e\left( \frac{j}{n} \right) \right) \oplus \left( \bigoplus_{j=a_n}^{\mu_n-1} e\left( \frac{j}{n} \right) \right),
\]
then \( \mu(A_n^\mu) = \mu_n \) and
\[
\mu(A_n^k) = \mu(A_n^{(k,n)}) = \left\lfloor ((k, n)\mu_n - 1)/n \right\rfloor + 1
\]
\[
= \min\{ \mu(B^k) : B \in \mathcal{L}(\mathcal{H}), \mu(B^\mu) = \mu_n \}
\]
for all \( k = 1, 2, \ldots \), where \( (k, n) = \text{G.C.D.}(k, n) \) and \( \mathcal{H} \) is an arbitrary complex separable Banach space with \( \dim \mathcal{H} \geq \mu_n \).
PROOF. It follows from (3) and (5) that
\[ \mu(A^n_h) = \mu(A_n^{kh}) = \mu_n \quad (h = 1, 2, \ldots) \]
and
\[ \mu(A_n) = \max\{s_n, s_n - 1\} = s_n. \]

Let \( Z_n = Z/nZ \). Observe that if \((k, n) = 1\), then the application “multiplication by \( k \)” is an automorphism of the ring \( Z_n \), whence we easily deduce (by using (5)) that \( \mu(A^k_n) = \mu(A_n) = s_n \). More generally, if \((k, n) = m, k = ma, n = mb\), then “multiplication by \( k \)” = (“multiplication by \( a \”)”∗ (“multiplication by \( m \””); the image of “multiplication by \( m \)” is \( mZ_n \cong Z_{(n/m)} = Z_b \), and “multiplication by \( a \)” is an automorphism of the subring \( mZ_n \) because \((a, b) = 1\). It follows from these observations, along with (4) and (5), that
\[ \mu(A^k_n) = \mu(A_n^m) \geq \left(\frac{m \mu_n - 1}{n}\right) + 1 \quad \text{for all } k = 1, 2, \ldots. \]

Thus, in order to complete the proof, we only have to show that if \( 1 < m < n \) and \( m | n \), then \( \mu(A_n^m) = \left[\frac{(m \mu_n - 1)}{n}\right] + 1 \). By using (5), we have
\[
\mu(A_n^m) = \max_{0 \leq i < n}\left\{s_n \left(c\{j: jm \equiv t \pmod{n}, 0 \leq j < a_n\}\right) + (s_n - 1)\left(c\{j: jm \equiv t \pmod{n}, a_n \leq j < n\}\right)\right\}
\]
\[
= \max_{0 \leq i < n}\left\{s_n \cdot c\{j: jm \equiv t \pmod{n}\} - c\{j: jm \equiv t \pmod{n}, a_n \leq j < n\}\right\}
\]
\[
= s_n \cdot c\{j: jm \equiv 0 \pmod{n}\} - c\{j: jm \equiv 0 \pmod{n}, a_n \leq j < n\}
\]
\[
= ms_n - c\{l: a_n \leq l(n/m) < n\} \quad \text{(using the fact that } \alpha_0 = s_n, \alpha_1 = s_n, \ldots, \alpha_n = s_n - 1\text{,}
\]
\[
= ms_n - c\{l: (m/n) a_n \leq l < m\} = ms_n - \left[m - \left(m a_n/n\right)\right]
\]
\[
= ms_n - \left[ms_n - (m \mu_n/n)\right]
\]
\[
= m\left[\left(\mu_n - 1\right)/n\right] + m - \left[m\left(\frac{\mu_n - 1}{n}\right) + m - (m \mu_n/n)\right] \quad \text{(using (6)).}
\]

Let \( \mu_n = hn + g \), where \( h \geq 0, 0 \leq g < n \); then a straightforward computation shows that
\[
(7) \quad \mu(A_n^m) = \begin{cases} mh & \text{if } g = 0, \\ mh + m - \left[m - (mg/n)\right] & \text{if } 1 \leq g < n. \end{cases}
\]

On the other hand,
\[
\left[(m \mu_n - 1)/n\right] + 1 = \begin{cases} mh & \text{if } g = 0, \\ mh + \left[(mg - 1)/n\right] + 1 & \text{if } 1 \leq g < n. \end{cases}
\]

Thus, \( \mu(A_n^m) = \left[(m \mu_n - 1)/n\right] + 1 = mh \) for the case when \( g = 0 \). If \( ln/m < g \leq (l + 1)(n/m) \), then
\[
\mu(A_n^m) = \left[(m \mu_n - 1)/n\right] = mh + l + 1, \quad l = 0, 1, 2, \ldots, m - 1,
\]
whence we conclude that \( \mu(A_n^m) = \left[(m \mu_n - 1)/n\right] + 1 \) for all \( m | n, 1 < m < n. \)

The proof of Lemma 3 is now complete. □
Proof of Theorem 1. Suppose that \( \mathcal{H} \) is a Hilbert space with orthonormal basis \( \{ e_i \}_{i=1}^{\infty} \). Define

\[
A_1 = 1^{(\mu_1)} \text{ on } V\{ e_1, e_2, \ldots, e_{\mu_1} \},
A_2 \text{ on } V\{ e_{\mu_1 + 1}, e_{\mu_1 + 2}, \ldots, e_{\mu_1 + \mu_2} \},
\vdots
A_n \text{ on } V\{ e_{\mu_1 + \mu_2 + \ldots + \mu_{n-1} + 1}, e_{\mu_1 + \mu_2 + \ldots + \mu_{n-1} + 2}, \ldots, e_{\mu_1 + \mu_2 + \ldots + \mu_n} \},
\vdots
\]

exactly as in Lemma 3.

Let \( \{ r_n \}_{n=1}^{\infty} \) be any strictly decreasing sequence of positive reals with \( r_1 = 1 \). Now we define

\[
T(M) = \bigoplus_{n=1}^{\infty} r_n A_n \in \mathcal{B}(\mathcal{H}).
\]

It is easy to check that \( T(M) \) is normal, \( ||T(M)|| = 1 \),

\[
\sigma(T(M)) \subset \left( \bigcup_{n=1}^{\infty} \left\{ r_n e^j \left( \frac{j}{n} \right) : j = 0, 1, 2, \ldots, n - 1 \right\} \right)^{-1},
\]

and

\[
\mu(T(M)^k) = \sup_n \mu(A_n^k) \quad (k = 1, 2, \ldots).
\]

By using (2) and Lemma 3, we deduce that

\[
\mu(A_n^k) = \mu(A_{n+1}^{k-1}) = \left( \left( (k, n) \mu_n - 1 \right) / n \right) + 1 \leq \mu_k = \mu(A_k^k)
\]

for all \( n = 1, 2, \ldots \), and therefore

\[
\mu(T(M)^k) = \mu_k \quad \text{for all } k = 1, 2, \ldots.
\]

Furthermore, if \( r_n \downarrow 0 \) fast enough, then \( T(M) \) is a nuclear operator. (It suffices to take \( r_n = (n + \mu_n)^{-1}, n = 1, 2, \ldots \).)

This proves Theorem 1 for the case when \( \mathcal{H} \) is a Hilbert space. If \( \mathcal{H} \) is not a Hilbert space, then it is enough to repeat the above construction with the orthonormal basis replaced by a normalized Markushevich basis (see [2]; the details are left to the reader). □

Proof of Theorem 2. We begin by constructing a nuclear normal operator \( T(M) \) exactly as in the previous proof.

Let \( L \) be a diagonal normal operator defined by \( Lf_{ij} = t_i e(v_j) f_{ij} \) with respect to an orthonormal basis \( \{ f_{ij} \}_{i,j=1}^{\infty} \), where \( \{ t_i \}_{i=1}^{\infty} \) is a denumerable dense subset of (distinct points of) \( (0,1) \setminus \{ r_n \}_{n=1}^{\infty} \) and \( \{ e(v_j) \}_{j=1}^{\infty} \) is a denumerable dense subset of the unit circle such that \( v_j \) and \( v_j / v_k \) are irrational for all \( j \) and, respectively, for all \( h \neq j \). Then \( L \) is a normal operator, the set of all eigenvalues of \( L^k \) is disjoint from the set of all eigenvalues of \( T(M)^k \) for each \( k = 1, 2, \ldots \), and it straightforward to check that

\[
\mu(\{ T(M) \oplus L \}^k) = \max \left\{ \mu(T(M)^k), \mu(L^k) \right\}
\]

\[
= \max \{ \mu_k, 1 \} = \mu_k \quad \text{for all } k = 1, 2, \ldots
\]
(see [1]), and

$$\sigma(T(M) \oplus L) = \sigma(L) = \{ \lambda : |\lambda| \leq 1 \}.$$ 

Thus, the normal operator \(N(M) = T(M) \oplus L\) satisfies all our requirements.

**References**


Department of Mathematics, Arizona State University, Tempe, Arizona 85287