SYMOMETRIC CUT LOCI IN RIEMANNIAN MANIFOLDS

W. VANNINI AND J. H. RUBINSTEIN

Abstract. Let $M$ be a compact Riemannian manifold with $H_1(M, \mathbb{Z}) = 0$. We show that, for a point $p \in M$, the cut locus and conjugate locus of $p$ must intersect if $M$ admits a group of isometries which fixes $p$ and has principal orbits of codimension at most 2. This is a classical theorem of Myers [5] in the case when $M$ has dimension 2.

0. In [5] Myers proved that if $M$ is a Riemannian manifold homeomorphic to $S^2$ and $p \in M$, then the cut locus and conjugate locus of $p$ in the tangent space $M_p$ must have a common point (also see Theorem 5.1 of [10]). On the other hand, Weinstein [10], answering a problem of Rauch [7], constructed a Riemannian metric on any compact simply-connected $C^\infty$ manifold not homeomorphic to $S^2$, so that there is a point $p \in M$ whose conjugate and cut loci are disjoint. The following conjecture was proposed by Weinstein [10]: "If $M$ is a compact simply-connected Riemannian manifold, then for some point $p \in M$, the conjugate locus and cut locus of $p$ intersect." Gromov has recently constructed metrics on $S^3$ with sectional curvature $\leq 1$ and arbitrarily small diameter, thus disproving this conjecture.

We give the following extension of Myers' result.

Theorem Suppose $M$ is a compact, connected, $C^\infty$ Riemannian manifold and there is a compact Lie group $G$ of isometries of $M$ which fix some point $p \in M$. Assume that $H_1(M, \mathbb{Z}) = 0$ and that a principal orbit of the $G$-action has codimension 2. Then the conjugate locus and cut locus of $p$ must have a point in common.

1. Remarks. (a) If $M$ has dimension 2, then, since $H_1(M, \mathbb{Z}) = 0$, it follows that $M$ is homeomorphic to $S^2$. If we take $G$ to be the trivial group, then the theorem becomes Myers’ result.

(b) All the 3-dimensional lens spaces $L(m, n)$ (see e.g. [6]) with the standard spherical metric admit $S^1$-actions which fix points $p$. Also the cut and conjugate loci of $p$ are disjoint, but $H_1(L(m, n), \mathbb{Z}) = \mathbb{Z}_m$.

(c) The Poincaré dodecahedral space $M^3$ (see [6]) with metric induced from $S^3$ is a homogeneous space admitting a transitive $SU(2)$-action. Moreover, $H_1(M, \mathbb{Z}) = 0$ and the cut and conjugate loci of any point are disjoint. However, the isotropy subgroup of any point is finite, so it has principal orbits of codimension 3.

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(d) In Berger’s classification [1] of normal Riemannian homogeneous spaces of strictly positive curvature, a class of Riemannian metrics on odd-dimensional spheres $S^{2n+1}$, of the form $SU(n+1) \times \mathbb{R}/SU(n) \times \mathbb{R}$, is given. It is easy to see that these examples satisfy the hypotheses of the theorem, and, hence, the conjugate and cut loci of any point must intersect. Note that the conjugate locus of a point in these manifolds is calculated in [3], and the cut locus, in the case $n = 1$, is computed in [8]. The result of the theorem applied to these examples of Berger for the case $n = 1$ is also given in [9].

2. Following Bredon [2], we introduce some transformation group notation. Let $M$ be a compact $C^\infty$ manifold, and let $G$ be a compact Lie group acting smoothly on $M$. The orbits $G_p$, $p \in M$, are partially ordered by the relation $G_p \leq G_q$ if the isotropy subgroup of $p$ is conjugate to a subgroup of the isotropy subgroup of $q$. A maximal orbit type is called a principal orbit, and the union of all principal orbits is labelled $U$.

The nonprincipal orbits are of two types. Let $d$ be the dimension of a principal orbit. Orbits with dimension strictly less than $d$ are called singular, while nonprincipal orbits with dimension $d$ are called exceptional. The union in $M$ of the singular (resp. exceptional) orbits is denoted by $B$ (resp. $E$).

Let $M^*$ denote the orbit space. If $S$ is a $G$-invariant set in $M$, let $S^*$ denote the projection of $S$ to $M^*$. Then $U$ (resp. $U^*$) is an open dense subset of $M$ (resp $M^*$). (See [2, Theorem 3.1, p. 179].) If $\dim M = n$ and $d = n - 1$ or $n - 2$, then $M^*$ is a manifold, possibly with boundary (cf. [2, Lemma 4.1, p. 186]).

With the notation of the theorem, let $C(p)$ (resp. $\tilde{C}(p)$) denote the cut locus of $p$ in $M$ (resp. $M_p$). Note that $\tilde{C}(p)$ is homeomorphic to $S^{n-1}$. The action of $G$ on $M$ can be lifted to a linear action of $G$ on the tangent space $M_p$. We let $\tilde{U}$ (resp. $\tilde{B}$, $\tilde{E}$) denote the union of the principal (resp. singular, exceptional) orbits in $M_p$. Finally, let $\tilde{D}(p)$ be the cell which is the closure of the bounded component of $M_p - \tilde{C}(p)$.

3. Proof of the Theorem. If $\dim M = 2$, the result follows by Myers’ theorem (cf. [5 and 10, Theorem 5.1]). Therefore we can assume $\dim M \geq 3$. Now the action of $G$ on $\tilde{D}(p)$ can be regarded as the cone of the action of $G$ on $\tilde{C}(p)$ with the origin as vertex, since $\tilde{D}(p)$ is star-like from the origin and the $G$-action on $M_p$ is linear. By [2, Theorem 8.2, p. 206], $\tilde{C}(p)^*$ is homeomorphic to either $S^1$ or $[0,1]$, since the principal orbits for the $G$-action on $\tilde{C}(p)$ have codimension one. In the former case $\tilde{C}(p)^*$ is a bundle over $S^1$, which gives a contradiction (by the homotopy sequence of a fibration applied to the $(n - 1)$-sphere $\tilde{C}(p)$). So $\tilde{D}(p)^*$ is a cone on the interval $\tilde{C}(p)^*$. Since $\dim M \geq 3$, $p$ is a singular orbit of the $G$-action on $M$, so $B^* \neq \emptyset$. Therefore, all the hypotheses of Theorem 8.6 in [2, p. 211] are satisfied for the $G$-action on $M$. We conclude that $E^* = \emptyset$, $M^*$ is a 2-disk with boundary $B^*$, and $\text{int} M^* = U^*$.

Let $\exp^*: M^*_p \to M^*$ be the map between orbit spaces induced by the $G$-equivariant map $\exp: M_p \to M$. Since $\exp: \tilde{D}(p) - \tilde{C}(p) \to M - C(p)$ is a diffeomorphism, it follows that $\exp^*: \tilde{D}(p)^* - \tilde{C}(p)^* \to M^* - C(p)^*$ is a homeomorphism.
Suppose that $\tilde{C}(p)$ has no conjugate points, i.e., $\exp$ is a local diffeomorphism at each point of $\tilde{D}(p)$. Hence, $G$-orbit dimensions are preserved by $\exp$ restricted to $\tilde{D}(p)$, and $\exp^*$ maps principal (resp. singular) orbits in $\tilde{D}(p)^*$ to principal (resp. singular) orbits in $M^*$. Furthermore, there are no exceptional orbits in $\tilde{D}(p)^*$ since $E^*$ is empty.

$\exp^*: \tilde{D}(p)^* \to M^*$ is a continuous map onto the 2-disk $M^*$. As above, $\tilde{D}(p)^* \cap \tilde{U}^* \to U^*$ and $\exp^*: \tilde{D}(p)^* \cap \tilde{B}^* \to B^*$. Also, $\text{int}(\tilde{D}(p)^*)$ must be contained in $\tilde{U}^*$ since it is mapped into $\text{int} M^* = U^*$ by $\exp^*$. We conclude that $\text{int} \tilde{C}(p)^* \subset \tilde{U}^*$ also, because $\tilde{D}(p)^*$ is a cone on the interval $\tilde{C}(p)^*$. The same reasoning shows that $\partial \tilde{D}(p)^* - \text{int} \tilde{C}(p)^* \subset B^*$. Note that $\exp^*$ projects the interval $\partial \tilde{D}(p)^* - \text{int} \tilde{C}(p)^*$ onto the circle $B^*$ by identifying the two endpoints of the interval.

We need to establish that $\exp^*$ is locally one-to-one on the arc $\tilde{C}(p)^*$. Suppose this is not the case. Then there are points $x, y, z$ in $\tilde{C}(p)$ with $Gy \neq Gz$, $\exp y = \exp z$, and elements $g, h \in G$, so that $g_i y_i \to x$ and $h_i z_i \to x$ as $i \to \infty$. Since $G$ is compact, by choosing subsequences it suffices to assume that $g_i \to g$ and $h_i \to h$ as $i \to \infty$. If $g = h$ then $y_i \to z_i$ and $\exp$ is not one-to-one in a neighbourhood of $x$. This contradicts the hypothesis that there are no conjugate points in $\tilde{C}(p)$. Hence, $g \neq h$. Also, $y_i \to g^{-1}x$ and $z_i \to h^{-1}x$ as $i \to \infty$, so $\exp g^{-1}x = \exp h^{-1}x$, i.e., $\exp x = \exp gh^{-1}x = gh^{-1}\exp x$. This proves that $\exp x$ has a nontrivial isotropy subgroup and, hence, belongs to an exceptional orbit, contradicting $E^* = \emptyset$. Therefore, $\exp^*$ restricted to $\tilde{C}(p)^*$ is locally one-to-one.

To complete the proof of the Theorem, we apply a similar argument to Theorem 5.1 of [10] (cf. [5] also) to conclude that there is a contradiction, since $M^* - C(p)^*$ is connected but $\exp^*: \tilde{C}(p)^* \to M^*$ is locally one-to-one with image $C(p)^*$. ($C(p)^*$ must be a tree, and, hence, $\exp^*$ cannot be locally one-to-one at the preimage of a vertex of this tree in int $M^*$.)

4. For completeness we note the following simple result when there is a codimension-one isometry group fixing a point.

**Proposition.** Suppose $M^n$ is a compact, connected, $C^\infty$ Riemannian manifold, there is a compact Lie group $G$ of isometries of $M$ which fix $p \in M$, and the principal orbits have codimension one. Then either the conjugate and cut loci of $p$ intersect, or $M$ is diffeomorphic to $RP^n$.

**Proof.** Clearly $G$ acts transitively on $\tilde{C}(p)$, so $\tilde{C}(p) = \{ x \in M_p: \|x\| = k \}$ for some constant $k$. By Lemma 5.6 of [4] either every point of $C(p)$ is a conjugate point of $p$, or $\exp x = \exp y$ for $x, y \in C(p)$ if and only if $x = -y$. In the latter case $\exp: \tilde{D}(p) \to M$ gives a diffeomorphism $\phi: RP^n \to M$ by identification of $RP^n$ with $\tilde{D}(p)/\sim$, where $x \sim y$ if and only if $x = -y$ and $\|x\| = k$.

**References**


Department of Mathematics, University of Melbourne, Parkville, Victoria 3052, Australia

(Current address of J. H. Rubinstein)

Current address (W. Vannini): Department of Mathematics, SUNY at Stony Brook, Stony Brook, New York 11794