EQUIVARIANT $K$-THEORY
AND REPRESENTATIONS OF HECKE ALGEBRAS

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ABSTRACT. We construct some representations of the Hecke algebra of an affine Weyl group using equivariant $K$-theory and state a conjecture on some $q$-analogs of the Springer representations.

1. This paper contains a new construction of the principal series representations of the Hecke algebra of an affine Weyl group, in terms of the equivariant $K$-theory of a flag manifold. The formulas defining the simplest operators in these representations are $q$-analogs of operators considered earlier by Demazure [1]. The results in this paper were found during a visit to the Tata Institute of Fundamental Research, Bombay, in December 1983; I am grateful to D. N. Verma for some stimulating discussions.

2. We recall (cf. Segal [4]) that if $X$ is a compact topological space with a continuous action of a compact topological group $M'$, then the equivariant $K$-theory $K_{M'}(X)$ is defined as the Grothendieck group of the category whose objects are the $M'$-equivariant complex vector bundles on $X$ and the morphisms are $M'$-equivariant maps with locally constant rank. Then $K_{M'}(X)$ is naturally an $R_{M'}$-module where $R_{M'} = K_{M'}$ (point) is the representation ring of $M'$, i.e. the Grothendieck group of the category of finite dimensional continuous complex representations of $M'$.

3. We shall need a variant of this definition, in which $M'$ is replaced by a complex Lie group $M$ underlying a (not necessarily connected) reductive complex algebraic group. We assume that $M$ acts continuously on the compact topological space $X$, and we wish to define $K_M(X)$.

According to Mostow (see [2, Chapter XV]), $M$ has maximal compact subgroups, any two such are conjugate and any connected component of the normalizer in $M$ of a maximal compact subgroup $M'$, meets $M'$. We shall construct, for any two maximal compact subgroups $M'$ and $M''$ of $M$, a canonical isomorphism $\phi_{M',M''}: K_{M'}(X) \rightarrow K_{M''}(X)$ as follows. Choose $g \in M$ such that $gM''g^{-1} = M'$. If $E$ is an $M'$-equivariant vector bundle on $X$, we define a new vector bundle $g^*E$ on $X$ as the inverse image of $E$ under the map $X \rightarrow X$, $x \mapsto gx$. It is clear that $g^*E$ is naturally an $M''$-equivariant vector bundle on $X$ and $\phi_{M',M''}$ is defined by $E \mapsto g^*E$. To show that $\phi_{M',M''}$ is independent of the choice of $g$, we may assume that $M' = M''$ and that $g \in M$ normalizes $M'$; we must show that $g^*E \approx E$ as $M'$-equivariant vector bundles. Since the isomorphism class of $g^*E$ does not change when $g$ runs through a fixed connected component of the normalizer of $M'$, and since $M'$ meets each such component, we can further assume that $g \in M'$. In this

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case, the $M'$-equivariant structure of $E$ defines an isomorphism $E_x \xrightarrow{g} E_{gx} = (g^*E)_x$ for all $x \in X$ and hence an isomorphism $g^*E \approx E$ as desired.

The isomorphisms $\phi_{M',M''}$ have an obvious transitivity property. We may therefore define $K_M(X)$ to be $\lim K_{M'}(X)$ (limit over all maximal compact subgroups $M'$ of $M$, with respect to the isomorphisms $\phi_{M',M''}$). Then we have natural isomorphisms $K(X) \cong K_{M'}(X)$ for any maximal compact subgroup $M' \subset M$. It also follows that $K_M(X)$ is naturally an $R_M$-module where $R_M = K_M(\text{point})$ is the representation ring of $M$, i.e. the Grothendieck group of the category of finite dimensional complex algebraic representations of $M$. (Note that $R_M \cong R_{M'}$ for any maximal compact subgroup $M' \subset M$.)

4. We now consider a simple, simply connected complex algebraic group $G$ and $X = G/B$, where $B$ is a Borel subgroup of $G$. Then $M = G \times \mathbb{C}^*$ acts on $X$ as follows: $G$ acts by left translation and $\mathbb{C}^*$ acts trivially. We have $K_M(X) = K_G(X) \otimes R_{\mathbb{C}^*} = K_G(X) \otimes \mathbb{Z}[q,q^{-1}]$, where $q$ is the generator of $R_{\mathbb{C}^*}$ corresponding to the identity representation $\mathbb{C}^* \to \mathbb{C}^*$.

Let $T$ be a maximal torus in $B$, $W$ the Weyl group of $G$ with respect to $T$, $S$ the set of simple reflections in $W$ (with respect to $B$), $P$ the lattice of weights $T \to \mathbb{C}^*$, $R \subset P$ the set of roots and $R^+$ the set of positive roots (with respect to $B$). For each $s \in S$, let $P_s$ be the parabolic subgroup $B \cup BsB$ and let $\pi_s: X \to G/P_s$ be the natural map. There is a unique endomorphism

\begin{equation}
(4.1) \quad T_s: K_M(X) \to K_M(X)
\end{equation}

with the following property: if $E$ is an $M$-equivariant algebraic vector bundle on $X$, then

\begin{equation}
(4.2) \quad E + T_sE = \pi_s^*(\pi_s)_*(E) - \pi_s^*(\pi_s)_*(E \otimes \Omega^1_s),
\end{equation}

where $\Omega^1_s$ is the line bundle on $X$ of holomorphic differential 1-forms along the fibres of $\pi_s$, regarded as an $M$-equivariant bundle with the obvious action of $G$ and with the action of $\mathbb{C}^*$ given by scalar multiplication on each fibre of $\Omega^1_s$. Here $(\pi_s)_*(E)$ is the alternating sum of the higher direct images of $E$ under $\pi_s$ in the category of coherent sheaves; these higher direct images are again $M$-equivariant algebraic vector bundles on $G/P_s$, hence their alternating sum defines an element in $K_M(G/P_s)$.

For any weight $p \in P$, we define an endomorphism

\begin{equation}
(4.3) \quad \theta_p: K_M(X) \to K_M(X)
\end{equation}

by

\begin{equation}
(4.4) \quad \theta_p E = E \otimes L_p^*,
\end{equation}

where $L_p$ is the line bundle on $X = G/B$ associated to the homomorphism $B \to \mathbb{C}^*$ obtained by composing the projection $B \to T$ with $p:T \to \mathbb{C}^*$, and $L_p^*$ is the dual line bundle; we regard $L_p^*$ as an $M$-equivariant bundle with the obvious action of $G$ and with trivial action of $\mathbb{C}^*$.

5. The group structure on the lattice of weights $P$ will be written multiplicatively. The Weyl group $W$ acts naturally on $P$ ($w:p \to w(p)$) and we form the semidirect product $\tilde{W} = W \cdot P$ with $P$ normal and $w \cdot p = w(p) \cdot w$ ($w \in W$, $p \in P$).
Then $\tilde{W}$ contains the affine Weyl group as a subgroup of finite index. According to Bernstein (see [3, 4.4]) one can describe the Hecke algebra $\tilde{H}$ corresponding to $\tilde{W}$ as follows. It is an algebra over $\mathbb{Z}[q, q^{-1}]$ with generators $T_s$ ($s \in S$) and $\theta_p$ ($p \in P$) subject to the following relations:

$$\begin{align*}
(5.1) & \quad (T_s + 1)(T_s - q) = 0 \quad (s \in S), \\
(5.2) & \quad T_s T_t T_s \cdots = T_t T_s T_t \cdots \\
& \quad \text{(s \neq t \in S; both sides have m_{s,t} factors where m_{s,t} = order of st in W)}, \\
(5.3) & \quad \theta_p \theta_{p'} = \theta_{pp'} \quad (p, p' \in P), \\
(5.4) & \quad T_s \theta_p = \theta_p T_s \quad (s \in S, p \in P, sp = ps), \\
(5.5) & \quad T_s \theta_{s(p)} T_s = q \theta_p \quad (s \in S, p \in P, sps^{-1}p^{-1} = \alpha_s^{-1}).
\end{align*}$$

Here $\alpha_s \in \mathbb{R}^+ \subset P$ is the simple root corresponding to $s$. We can now state our main result.

**Theorem.** The endomorphisms $T_s$, $\theta_p$ of $K_M(X)$ defined in §4 give rise to a left $\tilde{H}$-module structure on $K_M(X)$. (The action of $\mathbb{Z}[q, q^{-1}] \subset \tilde{H}$ is defined to be the same as the restriction to $\mathbb{R}^*_M$ of the earlier action of $R_M$.) This $\tilde{H}$-module structure commutes with the $R_M$-module structure on $K_M(X)$.

The proof will be given in §8.1.

6. We now fix a unipotent element $u \in G$. Let $\phi: SL_2(\mathbb{C}) \to G$ be an algebraic homomorphism such that $u = \phi(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix})$ and let $D$ be the group of diagonal matrices in $SL_2(\mathbb{C})$. We define

$$M_u = \{(g, \lambda) \in G \times \mathbb{C}^* \mid g^{-1}ug = u^\lambda, \ g \in Z_G(\phi(D))\}.$$ 

(Note that any complex power of a unipotent element is well defined.) Then $M_u$ is a reductive algebraic subgroup of $G \times \mathbb{C}^*$; if $u = e$ then $M_u = M$ of §4. Let $X_u = \{g_1 B \in X \mid g_1^{-1} u g_1 B \in B\}$; it is a closed subvariety of $X = G/B$ with an action of $M_u$ given by $(g, \lambda): g_1 B \to g g_1 B$. We now fix a connected component $c$ of $M_u$ and let $M_{u,c}$ be the inverse image under $M_u \to M_u/M_{u,0}$ of the cyclic group in $M_u/M_{u,0}$ generated by the image of $c$. The $R_{M_{u,c}}$-module $K_{M_{u,c}}(X_u)$ is defined as in §2. Restriction of vector bundles gives rise to a homomorphism of $R_{M_{u,c}}$-modules

$$R_{M_{u,c}} \otimes_{R_M} K_M(X) \to K_{M_{u,c}}(X_u).$$

(We regard $R_{M_{u,c}}$ as an $R_M$-module, via the homomorphism $R_M \to R_{M_{u,c}}$ induced by the inclusion $M_{u,c} \subset M$.) Now let $\overline{R}_{M_{u,c}}$ be the ring $R_{M_{u,c}}/J$, where $J$ is the ideal of $R_{M_{u,c}}$ consisting of all $E \in R_{M_{u,c}}$ whose character is identically zero on $c$. Then from (6.1) we get a homomorphism of $\overline{R}_{M_{u,c}}$-modules

$$\overline{R}_{M_{u,c}} \otimes_{R_M} K_M(X) \to \overline{R}_{M_{u,c}} \otimes_{R_{M_{u,c}}} K_{M_{u,c}}(X_u).$$

We have a natural surjective homomorphism $M_{u,c} \to \mathbb{C}^*$, $(g, \lambda) \to \lambda$; it induces a ring homomorphism $R_{C^*} \to \overline{R}_{M_{u,c}}$ and we shall denote the image of $q \in R_{C^*}$ in $\overline{R}_{M_{u,c}}$ again by $q$. We can now state the following conjecture.
CONJECTURE. There exists a natural left $\tilde{H}$-module structure on
\[ \mathcal{R}_{M_u,c} \otimes_{R_{M_u,c}} K_{M_u,c}(X_u) \]
with the following properties:
(a) It is compatible under (6.2) with the $\tilde{H}$-module structure on
\[ \mathcal{R}_{M_u,c} \otimes_{R_M} K_M(X) \]
deduced by extension of scalars from the $\tilde{H}$-module structure on $K_M(X)$ described in the Theorem.
(b) The action of $\theta_p$ ($p \in P$) on
\[ \mathcal{R}_{M_u,c} \otimes_{R_{M_u,c}} K_{M_u,c}(X_u) \]
is by tensor product with the restriction of $L^*_p$ to $X_u$ (see (4.4)).
(c) It commutes with the $\mathcal{R}_{M_u,c}$-module structure, and $q \in \tilde{H}$ acts in the same way as $q \in \mathcal{R}_{M_u,c}$.

7. Assuming the conjecture, let us consider a semisimple element $s \in G$ such that $(s,q_0) \in M_u$ for some $q_0 \in \mathbb{C}^*$. Let $c$ be the component of $M_u$ containing $s$. Let
\[ \tilde{H}(q_0) = C \otimes_{\mathbb{Z}[q,q^{-1}]} \tilde{H}, \]
where $C$ is regarded as a $\mathbb{Z}[q,q^{-1}]$-module with $q$ acting as multiplication by $q_0$. Let $h_s: R_{M_u,c} \rightarrow C$ be the ring homomorphism defined by $E \mapsto \text{Tr}(s,E)$ ($E \in R_{M_u,c}$). This homomorphism factors through $\mathcal{R}_{M_u,c}$, since $s \in c$. Consider the tensor product
\[ F_{u,s} = C \otimes_{R_{M_u,c}} K_{M_u,c}(X_u), \]
where $C$ is regarded as a $R_{M_u,c}$-module via $h_s$. Note that $F_{u,s}$ is a finite dimensional $C$-vector space. (Indeed by a theorem of Segal [4], $K_{M_u,c}(X_u)$ is a finitely generated $R_{M_u,c}$-module.) The conjecture implies that there is a natural left $\tilde{H}(q_0)$-module structure on $F_{u,s}$. It is a $q$-analog of the $W$-representations of Springer, which were extended to $\tilde{W}$ by S. Kato (Nederl. Akad. Wetensch. Proc. Ser. A 86 (1983), 193–201).

8. Proof of the theorem. It is well known that the elements $L_p \in K_G(X)$ ($p \in P$) form a $\mathbb{Z}$-basis of $K_G(X)$. Thus $K_G(X)$ may be identified with the group ring $\mathbb{Z}[P]$. It is also known that under this identification the canonical ring homomorphism $R_G \rightarrow K_G(X)$ becomes the inclusion of the $W$-invariants $\mathbb{Z}[P]^W$ into $\mathbb{Z}[P]$. It follows that we may identify $K_M(X)$ with $\mathcal{R}_0 = \text{group ring of } P \text{ over } \mathbb{Z}[q,q^{-1}]$ and the canonical ring homomorphism $R_M \rightarrow K_M(X)$ with the inclusion $\mathcal{R}_0^W \hookrightarrow \mathcal{R}_0$. We shall denote by $\mathcal{R}$ the quotient field of $\mathcal{R}_0$. With these identifications, the map $T_s: K_M(X) \rightarrow K_M(X)$ becomes the $\mathbb{Z}[q,q^{-1}]$-linear map $T_s: \mathcal{R}_0 \rightarrow \mathcal{R}_0$ given by
\[ T_s(\lambda) = \frac{\lambda - s(\lambda)}{\alpha_s - 1} - q \frac{\lambda - s(\lambda)\alpha_s}{\alpha_s - 1} \quad (\lambda \in P). \]
Note that this is a priori an element of $\mathcal{R}$. But it is easily seen that it is actually in $\mathcal{R}_0$. (If we specialize $q$ to 1 this is just the action $\lambda \rightarrow s(\lambda)$ of $W$ on $P$. If we
specialize \( q \) to 0 we obtain essentially Demazure’s operator [1]. Thus our operator (8.1) is the simplest possible interpolation between these two special cases.

The map \( \theta_p: K_M(X) \to K_M(X) \) becomes with the previous identifications the \( \mathbb{Z}[q,q^{-1}] \)-linear map \( \theta_p: \mathcal{R}_0 \to \mathcal{R}_0 \) defined by

\[
\theta_p(\lambda) = \lambda p^{-1}.
\]

We must show that the endomorphisms (8.1), (8.2) of \( \mathcal{R}_0 \) verify the identities (5.1)–(5.5).

The identities (5.1), (5.3) and (5.4) are immediate. Now let \( s \neq t \) be two simple reflections in \( S \) and let \( (s, t) \) be the subgroup of \( W \) they generate. Let \( \Phi_{s,t} \) be the set of roots which are \( \mathbb{Z} \)-combinations of the simple roots \( \alpha_s, \alpha_t \) and let \( \Phi_{s,t}^+ \cup \Phi_{s,t}^- \) be its partition into positive and negative roots. Let \( \rho_{s,t} \) be the element of \( P \) such that

\[
\rho_{s,t}^2 = \prod_{\alpha \in \Phi_{s,t}^+} \alpha.
\]

Let

\[
\Psi^+ = \prod_{\alpha \in \Phi_{s,t}^+} (1 - q\alpha), \quad \Psi^- = \prod_{\alpha \in \Phi_{s,t}^-} (1 - q\alpha).
\]

For any element \( \xi \in \mathcal{R}_0 \) we define \( \text{Alt}_{s,t}(\xi) = \sum_{w \in (s,t)} (-1)^l(w)w(\xi) \), where \( l \) is the length function on \( W \).

We denote the product \( T_sT_tT_s \cdots (i \text{ factors}) \) by \( T_{s,i} \) and similarly we denote the product \( T_tT_sT_t \cdots (i \text{ factors}) \) by \( T_{t,i} \). With these notations we state the following identity which is verified by direct computation. (Here \( m = \text{order of } st \text{ in } W. \))

\[
(1 + \sum_{1 \leq i \leq m} T_{s,i} + \sum_{1 \leq i \leq m-1} T_{t,i})(\lambda) = \text{Alt}_{s,t}(\lambda \rho_{s,t} \Psi^-) \cdot (\text{Alt}_{s,t}(\rho_{s,t}))^{-1} \quad (\lambda \in P).
\]

The right-hand side of this identity is symmetric in \( s, t \). Hence so is the left-hand side. It follows that \( T_{s,m} = T_{t,m} \) and (5.2) follows. (Alternatively, one can use the following identity:

\[
(-q)^m + \sum_{1 \leq i \leq m} (-q)^{m-i}T_{s,i} + \sum_{1 \leq i \leq m-1} (-q)^{m-i}T_{t,i} \quad (\lambda) = \rho_{s,t}^{-1}\Psi^+ \text{Alt}_{s,t}(\lambda) \cdot (\text{Alt}_{s,t}(\rho_{s,t}))^{-1} \quad (\lambda \in P).)
\]

With the assumption of (5.5) we have for \( \lambda \in P \):

\[
T_s\theta_{s(p)}T_s(\lambda) = T_s \frac{\lambda - s(\lambda) - q(\lambda - s(\lambda)\alpha_s)}{(\alpha_s - 1)s(p)}
= \frac{1}{\alpha_s - 1} \left( \frac{\lambda - s(\lambda) - q(\lambda - s(\lambda)\alpha_s)}{\alpha_s - 1} \right) (1 - q)
= q\lambda p^{-1} = q\theta_p(\lambda).
\]
We now show that the operators $T_s, \theta_p: \mathcal{R}_0 \rightarrow \mathcal{R}_0$ are $\mathcal{R}_0^W$-linear. This is trivial for $\theta_p$. For $T_s$, we have the following multiplicative property (whose proof is trivial):
\[
T_s(f \cdot g) = (T_s f) \cdot g + s(f) \cdot T_s(g) - q s(f) \cdot g
\]
for any $f, g \in \mathcal{R}_0$. If $f = s(f)$, then clearly $T_s f = q f$ so that $T_s(f \cdot g) = f \cdot T_s(g)$. In particular, if $f \in \mathcal{R}_0^W$, then $T_s(f \cdot g) = f \cdot T_s(g)$ for all $g \in \mathcal{R}_0$ and all $s \in S$. This shows that $T_s$ is $\mathcal{R}_0^W$-linear.

9. Remarks. (a) By a theorem of Pittie, $\mathcal{R}_0$ is a free $\mathcal{R}_0^W$-module of rank $|W|$. (b) The formula (8.3) has the following interpretation. Let $P_{s,t}$ be the parabolic subgroup $\bigcup_{w \in (s,t)} BwB$ and let $\pi_{s,t}$ be the natural map $G/B \rightarrow G/P_{s,t}$. Let us define $T_w$ for $w \in W$ as $T_{s_1} T_{s_2} \cdots T_{s_r}$, where $s_1 s_2 \cdots s_r$ is a reduced expression for $W$. Then for any $M$-equivariant algebraic vector bundle $E$ on $X = G/B$, we have
\[
\sum_{w \in (s,t)} T_w E = \sum_{i} (-1)^i \pi_{s,t}^* (\pi_{s,t})_* (E \otimes \Omega_{s,t}^i),
\]
where $\Omega_{s,t}^i$ is the vector bundle on $X$ of holomorphic differential $i$-forms along the fibres of $\pi_{s,t}$ regarded as an $M$-equivariant bundle with the obvious action of $G$ and with the action of $C^*$ given by scalar multiplication by $z^i$ on each fibre. (Compare (4.2).)

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