A ZETA-FUNCTION ASSOCIATED WITH ZERO TERNARY FORMS

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ABSTRACT. Consider the zeta-function associated with zero ternary forms defined as

$$\xi(t) = \sum_{x} \frac{1}{|\det x|^{t}} \quad (\text{Re } t \geq 2),$$

where $x$ runs over all $\text{SL}_3(\mathbb{Z})$-inequivalent zero ternary forms. We shall approximate $\xi(t)$ by another zeta-function which we can compute explicitly. By the approximation, we see that $\xi(t)$ is very close to $2\zeta(2)\zeta(2)$ which gives the contribution of zero ternary forms to the dimension formula of Siegel's cusp forms of degree three (computing via Selberg Trace Formula) up to a constant multiple.

1. Introduction. For each pair of nonzero integers $s_{13}$ and $s_2$, we define $\Delta(s_{13}, s_2)$ to be the set of ternary forms

$$0 \quad 0 \quad s_{13}$$
$$s_{13} \quad s_2 \quad s_{23}$$

Let

$$\mathcal{P} = \left\{ U = \begin{bmatrix} 1 & u & v \\ 0 & 1 & u \\ 0 & 0 & 1 \end{bmatrix} | u, v, w \text{ integers} \right\}.$$

$\mathcal{P}$ operates on $\Delta(s_{13}, s_2)$ by the action $S \rightarrow ^tUSU$. Let $\mu(s_{13}, s_2)$ be the number of inequivalent representatives of $\Delta(s_{13}, s_2)$ under the operation of $\mathcal{P}$. Consider the zeta function $\xi(t)$ defined as

$$\xi(t) = \sum_{s_2 \neq 0} \sum_{s_{13} = 1}^\infty \frac{\mu(s_{13}, s_2)}{(s_2s_{13})^t}.$$

We shall prove

**Theorem A.** For $\text{Re } t \geq 2$, we have

$$\xi(t) = \left\{ \frac{5}{2} + \frac{3}{2} \left( \frac{2^{-2t+1} + 2^{-3t} - 2^{-5t+2}}{1 - 2^{-3t}} \right) \right\} \frac{\xi(t)\zeta(2t-1)\zeta(3t-1)}{\zeta(3t)},$$

where $\zeta(t)$ is the Riemann zeta-function.
2. The special case \( t = 2 \).

**Lemma 1.** Let \( \delta(s_{13}, s_2) \) be the subset of \( \Delta(s_{13}, s_2) \), defined by
\[
\begin{align*}
0 &< s_{23} < (s_{13}, s_2) = \text{g.c.d. of } s_{13} \text{ and } s_2, \\
0 &< s_3 < l(s_{23}),
\end{align*}
\]
where \( l(s_{23}) \) is the least positive integer in the set
\[
\begin{align*}
\mathcal{G} = \left\{ 2k \cdot \frac{s_{23}s_{13}}{(s_{13}, s_2)} + k^2 \cdot \frac{s_{13}s_2}{(s_{13}, s_2)^2} + 2ns_{13} | k, n \in \mathbb{Z} \right\}.
\end{align*}
\]
Then we have
1. for each \( S \in \Delta(s_{13}, s_2) \), there exists \( U \in \mathcal{P} \) such that \( ^{t}USU \in \delta(s_{13}, s_2) \); and
2. if \( S_1, S_2 \in \delta(s_{13}, s_2) \), \( U \in \mathcal{P} \) and \( S_1 = ^{t}US_2U \), then \( S_1 = S_2 \).

**Proof.** For
\[
U = \begin{bmatrix} 1 & m & n \\ 0 & 1 & p \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 0 & 0 & s_{13} \\ 0 & s_2 & s_{23} \\ s_{13} & s_{23} & s_3 \end{bmatrix}
\]
in \( \mathcal{P} \) and \( \Delta(s_{13}, s_2) \), respectively, we let
\[
^{t}USU = \begin{bmatrix} 0 & 0 & s_{13} \\ 0 & s_2 & s'_{23} \\ s_{13} & s'_{23} & s'_3 \end{bmatrix}.
\]
Then a simple calculation shows
\[
\begin{align*}
s'_{23} &= s_{23} + ms_{13} + ps_2, \\
s'_3 &= s_3 + 2ps_{23} + p^2s_2 + 2ns_{13}.
\end{align*}
\]
First we choose integers \( m, p \) so that \( 0 \leq s'_{23} < (s_{13}, s_2) \). Note that the integral solutions of the equation \( ms_{13} + ps_2 = 0 \) are given by
\[
p = \frac{ks_{13}}{(s_{13}, s_2)}, \quad m = \frac{-ks_2}{(s_{13}, s_2)}, \quad k \text{ an integer}.
\]
Substituting the value of \( p \) as above into \( s'_3 \), we get
\[
s'_3 = s_3 + 2k \cdot \frac{s_{23}s_{13}}{(s_{13}, s_2)} + k^2 \cdot \frac{s_{13}s_2}{(s_{13}, s_2)^2} + 2ns_{13}.
\]
When \( k \) and \( n \) range over all integers, the set \( \mathcal{G} \) is a principal ideal of \( \mathbb{Z} \). Hence we can choose \( s' \) as asserted. (2) is obvious.

**Remark 1.** The set \( k^2 \cdot \frac{s_{13}s_2}{(s_{13}, s_2)^2} \) is a multiple of \( s_{13} \). If we let \( \tilde{s}_{13} = s_{13}/(s_{13}, s_2) \) and \( \tilde{s}_2 = s_2/(s_{13}, s_2) \), then we have
\[
l(s_{23}) = \begin{cases} 
2(s_{23}\tilde{s}_{13}, \tilde{s}_{13}) & \text{if } \tilde{s}_2 \text{ is even and } s_{23} \neq 0, \\
(2s_{23}\tilde{s}_{13}, \tilde{s}_{13}) & \text{if } \tilde{s}_2 \text{ is odd and } s_{23} \neq 0, \\
2|s_{13}| & \text{if } s_{23} = 0.
\end{cases}
\]
Remark 2. The set $\delta(s_{13}, s_2)$ in Lemma 1 is a subset of $M(s_{13}, s_2)$ which consists of matrices in $\Delta(s_{13}, s_2)$ with $0 \leq s_{23} < (s_{13}, s_2)$ and $0 \leq s_3 < 2s_{13}$. However, $\delta(s_{13}, s_2) \neq M(s_{13}, s_2)$ in general as shown by the following example:

\[
\begin{bmatrix}
1 & -1 & -1 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 3 \\
3 & 2 & 1 \\
0 & 0 & 1
\end{bmatrix}
= \begin{bmatrix}
1 & -1 & -1 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix}
= \begin{bmatrix}
0 & 0 & 3 \\
0 & 3 & 2 \\
3 & 2 & 2
\end{bmatrix}.
\]

From this lemma, it is easy to see that $\mu(s_{13}, s_2) = \mu(s_{13}, -s_2) = \mu(-s_{13}, s_2) = \mu(-s_{13}, -s_2)$. Hence it suffices to consider the case when $s_{13}$ and $s_2$ are positive integers. Here are some particular values of $\mu(s_{13}, s_2)$ for $s_{13} = 1, 2, 3, 4, 5$.

- $\mu(1, s_2) = \begin{cases} 2 & \text{if } s_2 \text{ is even}, \\ 1 & \text{if } s_2 \text{ is odd}; \end{cases}$
- $\mu(2, s_2) = \begin{cases} 4 & \text{if } s_2 = 4n + 1, 4n + 3, \\ 6 & \text{if } s_2 = 4n + 2, 4n; \\ 3 & \text{if } s_2 = 6n + 1, 6n + 5, \\ 6 & \text{if } s_2 = 6n + 2, 6n + 4, \\ 5 & \text{if } s_2 = 6n + 3, \\ 10 & \text{if } s_2 = 6n; \end{cases}$
- $\mu(3, s_2) = \begin{cases} 8 & \text{if } s_2 \text{ is odd}, \\ 12 & \text{if } s_2 = 8n + 2, 8n + 6, \\ 16 & \text{otherwise}; \\ 5 & \text{if } s_2 = 10n + 1, 10n + 3, 10n + 7, 10n + 9, \\ 10 & \text{if } s_2 = 10n + 2, 10n + 4, 10n + 6, 10n + 8, \\ 9 & \text{if } s_2 = 10n + 5, \\ 18 & \text{if } s_2 = 10n. \end{cases}$

For each fixed positive integer $s_{13}$, we define

\[
(3) \quad \eta(s_{13}) = \sum_{s_2 \neq 0} \frac{\mu(s_{13}, s_2)}{s_{13}^4 s_2^2} \cdot \frac{1}{\xi(2)}.
\]

Lemma 2. For any positive integer $k$, we have

\[
(4) \quad \eta(2^k) = \frac{2^{k+1}}{2^{4k}} + \frac{2^k}{2^{4k+2}} + \ldots + \frac{2^k}{2^{6k}}.
\]

Proof. Since the values of $\mu(s_{13}, s_2)$ are computed via Lemma 1 as

\[
\mu(2^k, s_2) = \begin{cases} 2^{k+1} + m \cdot 2^k & \text{if } (s_2, 2^{k+1}) = 2^m, 0 \leq m \leq k, \\ 2^{k+1} + k \cdot 2^k & \text{if } (s_2, 2^{k+1}) = 2^{k+1}; \end{cases}
\]

it follows that

\[
\eta(2^k) = \sum_{s_2 = 1}^{\infty} \frac{\mu(2^k, s_2)}{2^{4k} s_2^2} \cdot \frac{1}{\xi(2)} = \frac{2^{k+1}}{2^{4k}} + \frac{2^k}{2^{4k+2}} + \ldots + \frac{2^k}{2^{6k}}.
\]
Lemma 3. If \( p \) is an odd prime and \( m \) is a positive integer, then

\[
\eta(p^m) = \frac{5}{4} \left[ \frac{p^m}{p^{4m}} + \frac{\phi(p^m)}{p^{4m+2}} + \cdots + \frac{\phi(p^m)}{p^{6m}} \right],
\]

where \( \phi(p^m) = p^{m-1}(p-1) \) is the Euler \( \phi \)-function.

Proof. Let \( 1 \leq n \leq m \) and \( (s_2, 2p^m) = 2p^n \). If \( l(s_2) \) is the integer defined in Lemma 1 for such \( s_2 \), then we have \( l(0) = 2p^m \) and \( l(kp^n) = 2p^{m-n+u} \) if \( (k, p) = 1 \) and \( u \) is a nonnegative integer. (The total numbers of such \( k \)'s is \( \phi(p^{n-u}) \).) Hence we get

\[
\mu(p^m, s_2) = 2\left[ p^m + n\phi(p^m) \right].
\]

For the case \( (s_2, 2p^m) = p^n \), we get \( \mu(p^m, s_2) = p^m + n\phi(p^m) \) in the same manner. Hence our lemma follows from the definition of \( \eta(p^m) \).

Theorem B.

\[
\xi(2) = \frac{65}{24} \cdot \frac{\xi(2)\xi(3)\xi(5)}{\xi(6)}.
\]

Proof. By the definition of \( \eta \) and Lemma 1, we have

\[
\eta(a) = \alpha \sum_{d|a} \frac{a\phi(d)}{a^4 d^3},
\]

where \( \alpha = 2 \) if \( a \) is even and \( \alpha = \frac{1}{2} \) if \( a \) is odd. Consequently, if \( m \) and \( n \) are relatively prime integers, then a direct calculation shows \( \eta(mn) = \frac{1}{2} \eta(m)\eta(n) \).

Let \( \tilde{\eta}(a) = \frac{1}{2} \eta(a) \). Then by the previous lemmas, we have the following properties for \( \tilde{\eta}(m) \):

\[
(1) \quad \tilde{\eta}(2^k) = \frac{8}{5} \left[ \frac{2^k}{2^{4k}} + \frac{2^{k-1}}{2^{4k+2}} + \cdots + \frac{2^{k-1}}{2^{6k}} \right],
\]

\[
(2) \quad \tilde{\eta}(p^k) = \frac{p^k}{p^{4k}} + \frac{\phi(p^k)}{p^{4k+2}} + \cdots + \frac{\phi(p^k)}{p^{6k}}
\]

if \( p \) is an odd prime,

\[
(3) \quad \tilde{\eta}(mn) = \tilde{\eta}(m)\tilde{\eta}(n) \quad \text{if} \quad m \quad \text{and} \quad n \quad \text{are relative prime integers}.
\]

Hence

\[
\sum_{m=1}^{\infty} \tilde{\eta}(m) = \prod_{p \text{ prime}} \left( 1 + \tilde{\eta}(p) + \tilde{\eta}(p^2) + \cdots + \tilde{\eta}(p^n) + \cdots \right).
\]

For odd prime \( p \), we have

\[
1 + \sum_{k=1}^{\infty} \tilde{\eta}(p^k) = \frac{(1 - p^{-6})}{(1 - p^{-3})(1 - p^{-5})}.
\]

For the special case \( p = 2 \), we have

\[
1 + \sum_{k=1}^{\infty} \tilde{\eta}(2^k) = \frac{13}{12} \cdot \frac{(1 - 2^{-6})}{(1 - 2^{-3})(1 - 2^{-5})}.
\]
Hence
\[ \xi(2) = 2 \xi(2) \cdot \sum_{m=1}^{\infty} \eta(m) = \frac{5}{2} \xi(2) \sum_{m=1}^{\infty} \eta(m) \]
\[ = \frac{65}{24} \xi(2) \prod_{p : \text{prime}} \frac{(1 - p^{-6})}{(1 - p^{-3})(1 - p^{-5})} \]
\[ = \frac{65}{24} \frac{\xi(2) \xi(3) \xi(5)}{\xi(6)}. \]

3. The general case. For each fixed positive integer \( s_{13} \), we define
\[ \eta_t(s_{13}) = \frac{1}{2 \xi(2t)} \sum_{s_2 \neq 0} \mu(s_{13}, s_2) \left( s_{13}^t s_2^t \right) \quad \text{Re } t \geq 1. \]

Then we have
1. \( \eta_t(1) = \frac{5}{4}, \)
2. \[ \eta_t(2^k) = \frac{2k}{2^kt} + \frac{2^{k-1}}{2(2k+2)t} + \cdots + \frac{2^{k-1}}{2^{2k+2}t}, \]
3. \[ \eta_t(p^k) = \frac{5}{4} \left( \frac{p^k}{p^{2kt}} + \frac{\phi(p^k)}{p^{(2k+2)t}} + \cdots + \frac{\phi(p^k)}{p^{3kt}} \right). \]

if \( p \) is an odd prime,
4. \( \eta_t(mn) = \frac{5}{4} \eta_t(m) \eta_t(n) \) if \( m \) and \( n \) are relative prime integers.

From the computation we carried out before, we get

THEOREM A. For \( \text{Re } t \geq 2 \), we have
\[ \xi(t) = \left[ \frac{5}{2} + \frac{3}{2} \cdot \frac{2^{-2rt+1} + 2^{-3rt} - 2^{-5rt+2}}{1 - 2^{-3rt}} \right] \cdot \frac{\xi(t) \xi(2t-1) \xi(3t-1)}{\xi(3t)}. \]

4. Application and remark. Let \( S \) be a \( 3 \times 3 \) integral symmetric matrix of rank 3. We call \( S \) a zero ternary form if \( S \) represents zero in rational integers; i.e. there exists a nonzero integral vector \( u = [u_1, u_2, u_3] \) such that \( uSu^t = 0 \). Hence there exists a unimodular integral matrix \( U \) such that [3]
\[ USU^t = \begin{bmatrix} 0 & 0 & s_{13} \\ 0 & s_2 & s_{23} \\ s_{13} & s_{23} & s_3 \end{bmatrix}. \]
Set
\[ \mathcal{A}: \text{the set of representatives of zero ternary forms under the operation of unimodular matrices of } \text{GL}_3(\mathbb{Z}) \text{ by the action } S \rightarrow USU^t, \]
\[ \mathcal{B}: \text{the set of representatives of } G = \bigcup_{s_2 \neq 0} \bigcup_{s_{13} = 1} \Delta(s_{13}, s_2) \]\nunder the operation of \( \mathcal{B} \).
Then for each $S \in \mathcal{A}$, there exists a unimodular $U$ in $GL_3(\mathbb{Z})$ such that $USU^t \in \mathcal{B}$. Hence we can approximate the series

$$\xi(t) = \sum_{x \in \mathcal{A}} \frac{1}{|\det x|^t}, \quad \Re t \geq 2,$$

by the series

$$\xi(t) = \sum_{x \in \mathcal{A}} \frac{1}{|\det x|^t} \quad (\Re t \geq 2)$$

$$= \sum_{s_2 \neq 0} \sum_{s_1 = 1}^{\infty} \frac{\mu(s_{13}, s_2)}{(s_{13}^2 s_2)^t}$$

which contains $\xi(t)$ as a subseries. If we use the approximate values of zeta-functions as

\[
\begin{align*}
\zeta(2) &= 1.6449341, & \zeta(3) &= 1.2020569, \\
\zeta(5) &= 1.0369297, & \zeta(6) &= 1.0173431,
\end{align*}
\]

it follows that

$$\frac{65}{48} \cdot \frac{\zeta(3)\zeta(5)}{\zeta(6)} = 1.0086268\zeta(2).$$

Hence it is possible that $\zeta(2) = 2\zeta(2)\zeta(2)$ (a formula which is hard to verify directly).

Note that the zeta-function $\xi(t)$ we defined here is a constant multiple of a subseries of $\xi_2(s, L)$ appearing in [2] (restricted $L$ to zero ternary forms). This tells us that a constant multiple (the constant is $2^{-6\pi^2} - 4$ by a direct computation from the Selberg Trace Formula) of $\xi(2)$ gives the contribution of ternary forms to the dimension formula of Siegel’s cusp forms of degree three with respect to $Sp(3, \mathbb{Z})$.

REFERENCES


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