A CHARACTERIZATION IN THE SPACE OF
CONVOLUTION OPERATORS

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ABSTRACT. We give a characterization of $\mathcal{C}^\infty$ elements in the space of convolution operators $\theta'$, which belong to the Schwartz space $\mathcal{S}$.

In the space of convolution operators $\theta'$, also referred to as the space of distributions which are rapidly decreasing at $\infty$, there are $\mathcal{C}^\infty$ elements which are not rapidly decreasing in the sense of the Schwartz space $\mathcal{S}$. For a preliminary discussion of the spaces $\theta'$ and its image $\theta_M$ under the Fourier transform, see Treves [5, pp. 314–321], or Schwartz [4, Chapter VII, §8].

The purpose of this note is to give a necessary and sufficient condition for $\mathcal{C}^\infty$ elements in $\theta'$ to belong to $\mathcal{S}$. Notation used here but not defined is standard; see Hörmander [1, 2].

Let $p(\xi) \in \mathcal{S}^m$ be any “constant coefficient” symbol, that is, $p(\xi) \in C^\infty(\mathbb{R}^n)$ and satisfies for each multi-index $\alpha$ the estimate

$$|D_\xi p(\xi)| \leq c_\alpha (1 + |\xi|)^{m-|\alpha|}, \quad \xi \in \mathbb{R}^n,$$

where $c_\alpha$ are constants. Let $\rho(D)$ be the pseudo-differential operator corresponding to the symbol $\rho(\xi)$. Then clearly, for $f \in \mathcal{S}'$, one has

$$\rho(D)f(\xi) = \rho(\xi)\hat{f}(\xi) \in \theta_M.$$

Thus $\rho(D)$ may be considered as a map $\rho(D) : \mathcal{S}' \to \theta'$ and is sometimes referred to as a Friedrichs operator. With this notation our main result becomes the following

**Theorem.** Let $f \in \theta' \cap C^\infty(\mathbb{R}^n)$. Then $f \in \mathcal{S}$ if and only if there exists a Friedrichs operator $\rho(D)$ such that $f = \rho(D)v$ for some distribution $v$ with compact support.

**Remark.** A weaker form of the Theorem, namely that if $\chi(\xi)$ is a $C^\infty$ function on $\mathbb{R}^n$, positively homogeneous of degree 0 for $|\xi| > 1$ and if $v \in \mathcal{S}'$ is such that $\chi(D)v$ is $C^\infty$, then $|\chi(\xi)\hat{v}(\xi)| = O(|\xi|^{-N})$ for all positive integers $N$, has been mentioned in Nirenberg [3, p. 42].

The proof of the Theorem is based on the following

**Lemma.** $\{\rho(D)v : v \in \mathcal{S}'\} \cap C^\infty(\mathbb{R}^n) \subset \mathcal{S}$. 

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PROOF. Suppose \( \rho(D)v \in C^\infty(\mathbb{R}^n) \) for some \( v \in \mathcal{E}', \ v \neq 0 \). Then we may write (see [2, Theorem 2.13, p. 149])

\[
\rho(D) = \rho_1 + \rho_2
\]

where \( \rho_1 \) is properly supported, that is, maps \( C^\infty_0 \) into itself and \( \mathcal{E}' \) into itself; \( \rho_2 \) has a \( C^\infty \) kernel and defines a map on \( \mathcal{E}' \to C^\infty(\mathbb{R}^n) \). This decomposition is defined by a decomposition of the Schwartz kernel \( K_\rho \) associated with \( \rho(D) \), in the form

\[
K_{\rho_1} = \theta K_\rho \quad \text{and} \quad K_{\rho_2} = (1 - \theta) K_\rho
\]

where \( \theta \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n) \) which takes values in \([0, 1]\), equals 1 in a neighbourhood of the diagonal of \( \mathbb{R}^n \times \mathbb{R}^n \) and is properly supported.

The expression for the kernel \( K_\rho \) is given by

\[
K_\rho(x, y) = \frac{(2\pi)^{-n}}{(x - y)^a} \int_{\mathbb{R}^n} e^{i(x - y) \cdot \xi} (-D_\xi)^\alpha \rho(\xi) d\xi, \quad x, y \in \mathbb{R}^n, \ x \neq y,
\]

where the multi-index \( \alpha \) is arbitrary but chosen so that the integral converges absolutely. Now by (2), we have

\[
\rho(D)v = \rho_1(v) + \rho_2(v) \Rightarrow \rho_1(v) \in C^\infty_0(\mathbb{R}^n).
\]

Our aim now is to prove that \( \rho_2(v) \in \mathcal{S}' \). To this end, we shall first establish the form of the \( C^\infty \) function \( \rho_2(v) \). If we choose \( f \in C^\infty_0(\mathbb{R}^n) \) such that \( f = 1 \) in a neighbourhood of \( \text{supp}(v) \), then (see [2, Definition 2.11, p. 148])

\[
\rho_2(v)(x) = \rho_2(fv)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} p_f(x, \xi) b(\xi) d\xi
\]

where \( p_f \) stands for a symbol of \( \rho_2 \) in \( \mathcal{S}^{-\infty} \) and is given by the expression

\[
e^{ix \cdot \xi} p_f(x) = \rho_2(fe^{i(\cdot, \xi)})(x)
\]

where \( e^{i(\cdot, \xi)} \) denotes for each \( \xi \) the function \( e^{ix \cdot \xi} \). Now expressing the action of \( \rho_2 \) in terms of its kernel, we have from (3) and (4)

\[
\rho_2(fe^{i(\cdot, \xi)})(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \left[ \frac{(1 - \theta)}{(x - y)^a} \int_{\mathbb{R}^n} e^{i(x - y) \cdot \xi} (-D_\xi)^\alpha \rho(t) dt \right] f(y) e^{iy \cdot \xi} dy.
\]

Noting that \( \theta(x, y)f(y) \) has compact support since \( \theta \) is properly supported and integrating by parts with respect to \( y \), we obtain, for arbitrary multi-indices \( \beta_1, \beta_2 \)

\[
\sup_{(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n} \left| \xi^{\beta_1} D_\xi^{\beta_2} \rho_2(fe^{i(\cdot, \xi)})(x) \right| < \infty
\]

when \( \alpha \) is chosen so that \( |\alpha| > (m + n + |\beta_1| + |\beta_2|) \). Now from (6) and (7) we have

\[
\rho_2(v)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} b(\xi) \rho_2(fe^{i(\cdot, \xi)})(x) d\xi
\]

where we note that \( b \) has polynomial growth, say of order \( N \), and the integral is absolutely convergent because of (9). Differentiating under the integral sign in (10), it is clear that for arbitrary multi-indices \( \mu, \nu \) we have

\[
\sup_{x \in \mathbb{R}^n} \left| x^{\mu} D_x^{\nu} \rho_2(v)(x) \right| < \infty
\]
when the multi-index $\alpha$ appearing in (8) is chosen large enough, that is,

$$|\alpha| > \max\{|\mu|, m + 2n + N + 1 + |\nu|\}.$$ 

Thus, $p_2(v) \in \mathcal{S}$. Hence by (5) $\rho(D)v \in \mathcal{S}$ and the proof of the Lemma is complete.

**Proof of the Theorem.** $f \in \mathcal{S} \iff \hat{f}(\xi) \in \mathcal{S} \iff \hat{f}(\xi) \in \mathcal{S}^m \forall$ real $m$. Also we have $f = \hat{f}(D)\delta$, where $\delta$ is the Dirac measure on $\mathbb{R}^n$. The rest of the proof follows from the Lemma.

**References**


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