AN EXTREMAL PROBLEM FOR POLYNOMIALS WITH NONNEGATIVE COEFFICIENTS

GRADIMIR V. MILOVANOVIĆ

Abstract. Let $W_n$ be the set of all algebraic polynomials of exact degree $n$ whose coefficients are all nonnegative. For the norm in $L^2[0, \infty)$ with generalized Laguerre weight function $w(x) = x^a e^{-x}$ ($a > -1$), the extremal problem $C_n(a) = \sup_{P \in W_n} \left( \| P' \| / \| P \| \right)^2$ is solved, which completes a result of A. K. Varma [7].

1. In this paper we give the complete solution of a problem which has been investigated recently by A. K. Varma (see [7, 8]). This problem is related to some previous integral inequalities of Varma [9, 10] and also to the classical inequalities of A. Markov [4], P. Erdős [1], G. G. Lorentz [2, 3], G. Szegö [5], and P. Turan [6].

Let $W_n$ be the set of all algebraic polynomials of exact degree $n$, all coefficients of which are nonnegative, i.e.,

$$W_n = \left\{ P_n | P_n(x) = \sum_{k=0}^{n} a_k x^k, a_k \geq 0 \ (k = 0, 1, \ldots, n) \right\}.$$ 

We denote by $W_n^0$ the subset of $W_n$ for which $a_0 = 0$ (i.e., $P_n(0) = 0$).

Let $w(x) = x^a e^{-x}$ ($a > -1$) be a weight function on $[0, \infty)$, and let $\| f \|^2 = (f, f)$, where

$$(f, g) = \int_{0}^{\infty} w(x) f(x) g(x) \, dx \quad (f, g \in L^2[0, \infty)).$$

In [7] Varma has investigated the problem of determining the best constant in the inequality

$$(1.1) \quad \| P_n' \|^2 \leq C_n(a) \| P_n \|^2,$$

where $P_n \in W_n$. In fact, he has proved

**Theorem A.** Let $P_n(x)$ be an algebraic polynomial of exact degree $n$ with nonnegative coefficients. Then for $\alpha \geq (\sqrt{5} - 1)/2$,

$$\int_{0}^{\infty} \left( P_n'(x) \right)^2 x^a e^{-x} \, dx \leq \frac{n^2}{(2n + \alpha)(2n + \alpha - 1)} \int_{0}^{\infty} P_n^2(x) x^a e^{-x} \, dx,$$

equality holding for $P_n(x) = x^n$. For $0 \leq \alpha \leq 1/2$ we have

$$(1.2) \quad \int_{0}^{\infty} \left( P_n'(x) \right)^2 x^a e^{-x} \, dx \leq \frac{1}{(2 + \alpha)(1 + \alpha)} \int_{0}^{\infty} P_n^2(x) x^a e^{-x} \, dx.$$

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Moreover, (1.2) is also best possible in the sense that for \( P_n(x) = x^n + \lambda x \) the expression on the left can be made arbitrarily close to the expression on the right by choosing \( \lambda \) positive and sufficiently large.

The case \( \alpha = 1 \) was considered in [8]. The cases \( \alpha \in (-1, 0) \) and \( \alpha \in (1/2, (\sqrt{5} - 1)/2) \) are still open.

2. The object of this paper is to determine

\[
(2.1) \quad C_n(\alpha) = \sup_{P_n \in W_n} \frac{\|P_n\|^2}{\|P_n\|^2}
\]

for all \( \alpha \in (-1, \infty) \) and, thus, to give a complete solution of the extremal problem (1.1). Note that the supremum in (2.1) is attained for some \( P_n \in W_n^0 \). Indeed,

\[
\sup_{P_n \in W_n} \frac{\|P_n\|^2}{\|P_n\|^2} = \sup_{P_n \in W_n^0} \frac{\|P_n\|^2}{\|P_n + a_0\|^2} = \sup_{P_n \in W_n^0} \frac{\|P_n\|^2}{\|P_n\|^2}.
\]

We begin by proving three lemmas:

**Lemma 1.** If \( P_n \in W_n \) then for every \( x \geq 0 \) the inequality

\[
(2.2) \quad x\left(P_n'(x)^2 - P_n(x)P_n''(x)\right) \leq P_n'(x)P_n(x)
\]

holds.

**Proof.** Let \( P_n \in W_n \), i.e., \( P_n(x) = \sum_{k=0}^{n} a_k x^k \) with \( a_k \geq 0 \) (\( k = 0, 1, \ldots, n \)). Using the Cauchy-Schwarz inequality

\[
\left( \sum_{k=0}^{n} x_k y_k \right)^2 \leq \left( \sum_{k=0}^{n} |x_k|^2 \right) \left( \sum_{k=0}^{n} |y_k|^2 \right)
\]

for \( x_k = a_k^{1/2} x^{k/2} \) and \( y_k = k a_k^{1/2} x^{k/2} (x \geq 0) \), we obtain

\[
\left( \sum_{k=0}^{n} k a_k x^k \right) \leq \left( \sum_{k=0}^{n} a_k x^k \right) \left( \sum_{k=0}^{n} k^2 a_k x^k \right),
\]

which is equivalent to (2.2). \( \square \)

**Lemma 2.** If \( P_n \in W_n^0 \), then for the integrals

\[
J_n(\alpha) = \int_{0}^{\infty} x^{\alpha} e^{-x} P_n'(x)^2 dx,
\]

\[
I_n(i)(\alpha) = \int_{0}^{\infty} x^{\alpha} e^{-x} P_n(x) P_n^{(i)}(x) dx \quad (i = 0, 1, 2)
\]

the following recurrence relations hold:

\[
I_{n,2}(\alpha) = I_{n,1}(\alpha) - \alpha I_{n,1}(\alpha - 1) - J_n(\alpha) \quad (\alpha > -1),
\]

\[
2I_{n,1}(\alpha) = I_{n,0}(\alpha) - \alpha I_{n,0}(\alpha - 1) \quad (\alpha > -2).
\]

The proof of this lemma is a simple application of integration by parts and will be omitted. We note that the integrals \( I_{n,1}(\alpha) \) and \( I_{n,0}(\alpha - 1) \) exist for \( \alpha > -2 \) because \( P_n(0) = 0 \).
From Lemmas 1 and 2 there immediately follows

**Lemma 3.** If $P_n \in W_n^0$, then for $\alpha > -1$,

$$ J_n(\alpha) \leq \frac{1}{4} \left\{ I_{n,0}(\alpha) + (1 - 2\alpha)I_{n,0}(\alpha - 1) + (\alpha - 1)^2I_{n,0}(\alpha - 2) \right\}. $$

**Theorem.** The best constant $C_n(\alpha)$ defined in (2.1) is

$$ C_n(\alpha) = \begin{cases} \frac{1}{2 + \alpha}(1 + \alpha) & (-1 < \alpha \leq \alpha_n), \\ \frac{n^2}{2n + \alpha}(2n + \alpha - 1) & (\alpha_n \leq \alpha < +\infty), \end{cases} $$

where

$$ \alpha_n = \frac{1}{2}(n + 1)^{-1}\left[(17n^2 + 2n + 1)^{1/2} - 3n + 1\right]. $$

**Proof.** Let $P_n \in W_n^0$, i.e., $P_n(x) = \sum_{k=1}^n a_k x^k (a_k \neq 0)$. Then

$$ P_n(x)^2 = \sum_{k=2}^{2n} b_k x^k \quad (b_k \neq 0) $$

and

$$ \|P_n\|^2 = I_{n,0}(\alpha) = \sum_{k=2}^{2n} b_k \Gamma(k + \alpha + 1), $$

where $\Gamma$ is the gamma function. Using Lemma 3 we obtain

$$ 4J_n(\alpha) \leq \sum_{k=2}^{2n} b_k \left\{ \Gamma(k + \alpha + 1) + (1 - 2\alpha)\Gamma(k + \alpha) + (\alpha - 1)^2\Gamma(k + \alpha - 1) \right\}, $$

i.e.,

$$ J_n(\alpha) \leq \sum_{k=2}^{2n} H_k(\alpha) b_k \Gamma(k + \alpha + 1), $$

where

$$ H_k(\alpha) = \frac{1}{4} \cdot k^2/(k + \alpha)(k + \alpha - 1). $$

From (2.5) it follows that

$$ \|P_n\|^2 \leq \left( \max_{2 \leq k \leq 2n} H_k(\alpha) \right)\|P_n\|^2, $$

so

$$ C_n(\alpha) \leq \max_{2 \leq k \leq 2n} H_k(\alpha). $$

Determining the maximum of $f(x) = x^2/(x + \alpha)(x + \alpha - 1)$ on the interval $[2, 2n]$, we find that

$$ \max_{2 \leq k \leq 2n} H_k(\alpha) = \begin{cases} H_2(\alpha) & \text{if } -1 < \alpha \leq \alpha_n, \\ H_{2n}(\alpha) & \text{if } \alpha_n \leq \alpha < +\infty, \end{cases} $$

where $\alpha_n$ is given by (2.4).
In order to show that \( C_n(\alpha) \) defined in (2.3) is best possible, i.e. that \( C_n(\alpha) = \max_{2 \leq k \leq 2n} H_k(\alpha) \), we consider \( \hat{P}_n(x) = x^n + \lambda x \ (\lambda \geq 0) \) and set
\[
Q_n(\lambda) = \frac{\|\hat{P}_n\|^2}{\|\hat{P}_n\|^2}.
\]
By a simple computation we find that
\[
Q_n(\lambda) = \frac{n^2 \Gamma(2n + \alpha - 1) + 2\lambda n \Gamma(n + \alpha)}{\Gamma(2n + \alpha + 1) + 2\lambda \Gamma(n + \alpha + 2)}.
\]
Since
\[
Q_n(0) = \frac{n^2}{(2n + \alpha)(2n + \alpha - 1)} = H_z(\alpha)
\]
and
\[
\lim_{\lambda \to \infty} Q_n(\lambda) = \frac{1}{(\alpha + 1)(\alpha + 2)} = H_2(\alpha),
\]
we conclude that \( \hat{P}_n(x) = x^n \) is an extremal polynomial for \( \alpha \geq \alpha_n \). Furthermore, if \(-1 < \alpha \leq \alpha_n \), there exists a sequence of polynomials, for example, \( p_{n,k}(x) = x^n + kx \), \( k = 1, 2, \ldots \), for which
\[
\lim_{k \to \infty} \frac{\|p'_{n,k}\|^2}{\|p_{n,k}\|^2} = C_n(\alpha). \quad \Box
\]

**Remark.** From (2.4) we have \( \alpha_1 = (\sqrt{5} - 1)/2 \), \( \alpha_2 = (\sqrt{73} - 5)/6 \), \( \alpha_3 = (\sqrt{10} - 2)/2 \), etc. The sequence \( (\alpha_k) \) is decreasing, i.e., \( \alpha_1 > \alpha_2 > \alpha_3 > \cdots > \alpha_\infty \), where \( \alpha_\infty = \lim_{n \to \infty} \alpha_n = (\sqrt{17} - 3)/2 \approx 0.56155 \).

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**References**


Faculty of Electronic Engineering, Department of Mathematics, University of Niš, Beogradska 14, P. O. Box 73 18000 Niš, Yugoslavia