ALMOST EUCLIDEAN QUOTIENT SPACES OF SUBSPACES
OF A FINITE-DIMENSIONAL NORMED SPACE

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Abstract. The main result of this article is Theorem 1 which states that a quotient
space \( Y, \dim Y = k \), of a subspace of any finite dimensional normed space \( X, \dim X = n \), may be chosen to be \( d \)-isomorphic to a euclidean space even for
\( k = [\lambda n] \) for any fixed \( \lambda < 1 \) (and \( d \) depending on \( \lambda \) only).

The following theorem is proved.

1. Theorem. For every \( d > 1 \) there exists \( \lambda(d) > 0 \) such that every \( n \)-dimensional
normed space \( X \) contains a \( k \)-dimensional quotient space \( F \) of a subspace \( E \subset X \) which
satisfies

\[
\begin{align*}
(i) & \quad d(F, l^2_k) \leq d, \\
(ii) & \quad \dim F = k \geq \lambda(d)n.
\end{align*}
\]

(Here \( d(F, l^2_k) \) denotes a Banach-Mazur distance between two normed spaces; i.e.,
\( d(X, Y) = \inf \{ \| T \| \cdot \| T^{-1} \| \text{ over all linear isomorphisms } T: X \to Y \} \).

Moreover, \( \lambda(d) \to 1 \) if \( d \to \infty \) and, for large \( d \), \( \lambda(d) = 1 - 3\sqrt{6} /\ln \ln d \).

Remark 1. It is enough to prove Theorem 1 for large \( d \) only, because, as proved in
[MJ], any \( d \)-isomorphic copy of \( l^2_k \) contains, for any \( \epsilon > 0 \), a \((1 + \epsilon)\)-isomorphic
copy of \( l^2_k \), where \( k \geq \kappa(\epsilon)m/d^2 \) and \( \kappa(\epsilon) > 0 \) depends on \( \epsilon > 0 \) only.

Remark 2. Of course, the theorem states that the dual \( E^* \) to \( E \subset X \) contains a
subspace \( F^* \subset E^* \) which satisfies (i) and (ii) of the theorem.

Remark 3. In [M2] the theorem was proved with a logarithmic factor, and this
theorem was formulated as a problem. We refer the reader to this paper for relevant
discussion.

2. Notations. Let \( X \) be an \( n \)-dimensional normed space, i.e., \( R^n \) with the norm \( \| \cdot \| \),
and let \((x, y)\) be an inner product on \( X \); consequently, \( |x| = (x, x)^{1/2} \) is a euclidean
norm on \( X \). For any \( x \in X \) let \( (1/a)|x| \leq \|x\| \leq b|x| \) and \( M_r = \int_{S^{n-1}} \|x\| d\mu(x), \)
where \( S^{n-1} = \{ x \in X : |x| = 1 \} \) and \( \mu(x) = \mu_{n-1}(x) \) is the normalized invariant
(Haar) measure on \( S^{n-1} \). Let \( \|x\|^* = \sup_{y \neq 0} ((x, y)\|y\|) \). Then \((1/b)|x| \leq \|x\|^* \leq
a|x| \), and we define \( M_{r*} = \int_{S^{n-1}} \|x\|^* d\mu(x). \)

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Let \( K = \{ x \in X: ||x|| \leq 1 \} \), \( K^* = \{ x \in X: ||x||^* \leq 1 \} \), and \( D = \{ x \in X: |x| \leq 1 \} \). We consider also the usual \((n\text{-dimensional})\) Lebesgue measure \((\text{Vol}_n)\) on \(R^n\) normalized (for example) so that the induced measure on \(S^{n-1}\) coincides with \(\mu(x)\); that is,

\[ \text{Vol}_n(D) = (1/n)\mu(S^{n-1}) = 1/n. \]

We will use the following geometrical inequalities:

1. \((\text{Vol}_n K/\text{Vol}_n D)^{1/n} \leq M_{\ast}^*\quad\text{(the Urysohn inequality [U])},\]
2. \(\text{Vol}_n K \cdot \text{Vol}_n K^* \leq (\text{Vol}_n D)^2\quad\text{(the Santalo inequality [S])}\).

Also define \(M_{\ast} \cdot M_{\ast}^* = M\).

3. We prove the following proposition (see also [M2]).

**Proposition.** For every \(0 < \lambda < 1\), there exists a subspace \(E \subset X\), \(\dim E \geq \lambda n\), such that \(E^*\) contains a subspace \(F \subset E^*\), \(\dim F = k \geq \lambda^2 n\), such that

\[ d(F, l_k^1) \leq \left[ C_1(M + 1) \right]^{(1/\lambda)^2}, \]
where \(C_1\) is an absolute constant (say \(\gamma \sim 8\pi\)).

**Proof.** We start with a general argument valid for an arbitrary euclidean norm \(\cdot\) on \(R^n\). This norm will be defined in §§5 and 6. We introduce an additional norm \(\text{\| \cdot \|}_1\) on \(X\) such that

\[ K_1 = \{ x \in X: \|x\|_1 \leq 1 \} = \text{Conv}(K \cup D). \]

Then \(K_1^* = \{ x \in X: \|x\|_1^* \leq 1 \} = K^* \cap D\) (i.e., \(\|x\|_1^* = \max(\|x\|^*, |x|)\)). Therefore,

\[ M_{\ast}^* = \int_{S^{n-1}} \|x\|_1^* \, d\mu(x) \leq M_{\ast} + 1. \]

Since \(\|x\|_1 \leq |x|\), the so-called volume ratio of the pair \((K_1; D)\) is

\[ \nu(K_1) = (\text{Vol}_n K_1/\text{Vol}_n D)^{1/n} \leq M_{\ast} + 1 = A \]
(by (1)). Next we use the following statement, which is an immediate consequence of the technique of Szarek’s proof [Sz] of Kashin’s theorem [K] (for details see [M2]).

**4. Statement.** Let \(\nu(K_1) \leq A\). Fix \(0 < \lambda < 1\). Then for any \(k \leq \lambda n\) there exists a subspace \(E\), \(\dim E = k\), such that

\[ \frac{1}{2} (2\pi A)^{-\theta} |x| \|x\|_1 \leq \|x\|, \]
where \(\theta = 1/(1 - \lambda)\). The normalized Haar measure \(\nu_{n,k}\) of such subspaces in the Grassmann manifold \(G_{n,k}\) of \(k\)-dimensional subspaces of \(R^n\) is at least \(1 - 1/2^{n-1}\).

5. We return to the proof of Proposition 3 and consider the normalization \(M_{\ast} = 1\).

Let \(a = \{ x \in S: \|x\| \leq 2 \}\). Then obviously \(\mu(a) \geq 1/2\). Fix \(\delta > 0\). Let \(B = \{ \xi \in G_{n,k}: \mu_\xi(a \cap \xi) \geq \delta \}\). Again, obviously, \(\nu_{n,k}(B) = \gamma \geq \left(\frac{1}{2} - \delta\right)/(1 - \delta)\). Then for any \(\xi \in B\),

\[ \frac{\text{Vol}_\xi(K \cap \xi)}{\text{Vol}_\xi(D \cap \xi)} = \int_{S \cap \xi} \frac{1}{\|x\|^k} \, d\mu(x) \geq \frac{1}{2^k} \delta. \]

Choose \(\delta = 1/4\) and so \(\gamma \geq 1/3\).
Therefore, there exists a subspace $E_0$ (and actually a large measure of subspaces) as in Statement 4 such that
\begin{equation}
\Vol_k(K \cap E_0)/\Vol_k D(E_0) \geq 1/2^k \cdot 1/4,
\end{equation}
where $k = \dim E_0$. Consider now $E_0^*$. Then for any $x \in E_0^*$, $\|x\| \leq 2(2\pi A)^{\theta}|x|$, and by (3) and the Santalo inequality (2),
\begin{equation}
\Vol_k(K \cap E_0)^*/\Vol_k D(E_0) \leq 4 \cdot 2^k.
\end{equation}
Therefore, introducing a new norm on $E_0$: $\|x\|_2 = \|x\|/2(2\pi A)^{\theta}$, we have that $K_2 = \{x: \|x\|_2 \leq 1\}$ has the volume ratio
\[\text{vr}(K_2) \leq A_1 = 4(2\pi A)^{\theta} \cdot 4^{1/k} \quad \text{(and } \|x\|_2 \leq |x|).\]
So we may use Statement 4 one more time for the norm $\| \cdot \|_2$ to finish the proof of Proposition 3.

6. Proposition 3 contains a number $M$ which depends on the choice of a euclidean norm in $\mathbb{R}^n$. It is known [F, T.] that for every $(X, \| \cdot \|)$ there exists a euclidean norm $\| \cdot \|$ such that $M + 1 \leq c_2\|\text{Rad}_X\|$, where $c_2$ is an absolute constant and $\|\text{Rad}_X\|$ is the norm of the so-called Rademacher projection of $L_2(X)$ onto $\text{Rad} X$, which, as G. Pisier [P] has proved, may be estimated by
\begin{equation}
\|\text{Rad}_X\| \leq c_3\ln(d_X + 1),
\end{equation}
where $c_3$ is an absolute constant and $d_X = d(X, l_2^n) \leq \sqrt{n}$. Therefore, in particular,
\[\|\text{Rad}_X\| \leq c_3\ln(n + 1).
\]

7. Using 6, we may write in Proposition 3 that
\begin{equation}
d(F, l_2^n) \leq [c\ln(d_X + 1)]^{1/(1-\lambda)} \leq [c\ln(n + 1)]^{1/(1-\lambda)},
\end{equation}
where $c$ is a universal constant. We now use Proposition 3 and (5) consecutively many times starting with $\lambda_1$, obtaining a space $F_1$ (as in Proposition 3), $\dim F_1 = k_1 \geq \lambda_1^2 n$ with
\[d_1 = d(F_1, l_2^n) \leq [c\ln(n + 1)]^{1/(1-\lambda)}.
\]
For the second step we apply the same Proposition 3 to space $F_1$ (instead of $X$) with $\lambda_2$ and obtain a space $F_2$, $\dim F_2 = k_2 \geq (\lambda_2 \lambda_1)^2 n$ with
\[d_2 = d(F_2, l_2^n) \leq [c\ln(d_1 + 1)]^{1/(1-\lambda)},
\]
and so on.

It remains to state how we choose $\lambda_t$, $t = 1, 2, \ldots$. The notations $\ln^{(t)} A$ will be used for the $t$-times iterated logarithm of $A$ (so $\ln^{(2)} A = \ln \ln A$) if for any $k \leq t$, the $k$-iterated logarithm of $A$ is at least 2 and just 2 in the opposite case. With such an agreement we write, in (5), $\ln^{(1)} d_X = \ln d_X$ and $\ln^{(1)} n$ instead of $\ln(d_X + 1)$ and $\ln(n + 1)$.

For every $t \geq 1$, take $\lambda_t = 1 - \sqrt{6}/\ln^{(t+1)} n$, and we obtain, by using Proposition 3 $t$-times, a space $F_t$,
\begin{equation}
\dim F_t = k_t \geq \prod_{i=1}^t (1 - \sqrt{6}/\ln^{(t+1)} n)^2 n
\end{equation}
and
\[ d_t = d(F_t, l^2_{d_t}) \leq (c \ln d_{t-1})^{(\ln(t+1)n)^2/6}. \]

We assume now that \( c < \ln d_{t-1} \), and we just stop our iteration in the opposite case, \( d_{t-1} \leq e^c \). Therefore,
\[ d_t \leq (\ln d_{t-1})^{(\ln(t+1)n)^2/3} \]
and
\[ \ln d_t \leq \left( \left( \ln(t+1)n \right)^2/3 \right) \ln(2) d_{t-1}. \]

Now,
\[ \ln d_t \leq \left( \left( \ln^2 n \right)^3/3 \right) \leq \left( \ln^2 n \right)^3, \quad \ln d_2 \leq \left( \ln^3 n \right)^3, \]
and, in general,
\[ \ln d_t \leq \left( \ln^{(t+1)n} n \right)^3. \]

Now take \( d \) from the statement of Theorem 1. By Remark 1 we may assume \( d \) large enough; so let \( d > e^c \) and \( a = (\ln \ln d)/3 > \sqrt{6} \cdot 2 \). We stop our iteration procedure for \( t \) such that for the last time, \( \ln^{(t+1)n} n \geq a \) (i.e., \( e^a > \ln^{(t+1)n} n \)), which implies, by (8) and (7), \( d_t \leq \exp(e^a) = d \). Of course, the iteration could have stopped before because of the first condition if, for some \( j < t \), \( d_j \leq e^c < d \). Therefore, in both cases we have found a space \( F_j, j < t \), such that \( d(F_j, l^2_{d_j}) \leq d \). It remains to estimate \( \dim F_j = k \), using (6). On the last step of the iteration we have \( \ln^{(t+1)n} n \geq a ( > 2\sqrt{6} ) \), and, therefore, \( \lambda_t \geq (1 - \sqrt{6} /a), \lambda_{t-1} \geq (1 - \sqrt{6} /e^a) \), and so on; so it is enough to estimate from below the infinite product
\[ (1 - \sqrt{6} /a) \cdot (1 - \sqrt{6} /e^a) \cdot \ldots \cdot (1 - \sqrt{6} /e^{2a}) \cdots \geq \prod_{t=0}^\infty \left( 1 - \sqrt{6} /a^{2^t} \right) = f(a) \to 1 \]
if \( a \to \infty \), and, therefore, we prove the principal part of Theorem 1. It is also clear that the main part of this product is the first term \( f(a) = 1 - \sqrt{6} /a = 1 - 3\sqrt{6} /\ln \ln d \).

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Added in proof. I have now obtained the following better estimate for the function \( \lambda(d) \) in Theorem 1:
\[ \lambda(d) \geq 1 - c\sqrt{\frac{\log d}{d}} \quad \text{for large } d. \]

References


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