ALMOST EUCLIDEAN QUOTIENT SPACES OF SUBSPACES OF A FINITE-DIMENSIONAL NORMED SPACE

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Abstract. The main result of this article is Theorem 1 which states that a quotient space \( Y, \dim Y = k \), of a subspace of any finite dimensional normed space \( X, \dim X = n \), may be chosen to be \( d \)-isomorphic to an Euclidean space even for \( k = \lfloor \lambda n \rfloor \) for any fixed \( \lambda < 1 \) (and \( d \) depending on \( \lambda \) only).

The following theorem is proved.

1. Theorem. For every \( d > 1 \) there exists \( \lambda(d) > 0 \) such that every \( n \)-dimensional normed space \( X \) contains a \( k \)-dimensional quotient space \( F \) of a subspace \( E \subset X \) which satisfies

\[
\begin{align*}
(i) & \quad d(F, l_2^k) \leq d, \\
(ii) & \quad \dim F = k \geq \lambda(d)n.
\end{align*}
\]

(Here \( d(F, l_2^k) \) denotes a Banach-Mazur distance between two normed spaces; i.e.,
\[
\begin{align*}
d(X, Y) &= \inf \{ \|T\| \cdot \|T^{-1}\| \over \text{over all linear isomorphisms } T : X \to Y \}.
\end{align*}
\]
Moreover, \( \lambda(d) \to 1 \) if \( d \to \infty \) and, for large \( d \), \( \lambda(d) = 1 - 3\sqrt{6} / \ln \ln d \).)

Remark 1. It is enough to prove Theorem 1 for large \( d \) only, because, as proved in [M1], any \( d \)-isomorphic copy of \( l_2^n \) contains, for any \( \varepsilon > 0 \), a \( (1 + \varepsilon) \)-isomorphic copy of \( l_2^k \), where \( k \geq \kappa(\varepsilon)m/d^2 \) and \( \kappa(\varepsilon) > 0 \) depends on \( \varepsilon > 0 \) only.

Remark 2. Of course, the theorem states that the dual \( E^* \) to \( E \subset X \) contains a subspace \( F^* \subset E^* \) which satisfies (i) and (ii) of the theorem.

Remark 3. In [M2] the theorem was proved with a logarithmic factor, and this theorem was formulated as a problem. We refer the reader to this paper for relevant discussion.

2. Notations. Let \( X \) be an \( n \)-dimensional normed space, i.e., \( R^n \) with the norm \( \| \cdot \| \), and let \( (x, y) \) be an inner product on \( X \); consequently, \( |x| = (x, x)^{1/2} \) is a Euclidean norm on \( X \). For any \( x \in X \) let \( (1/a)|x| \leq \|x\| \leq b|x| \) and \( M_r = \int_{x \in S^{n-1}} \|x\| \, d\mu(x) \), where \( S^{n-1} = \{ x \in X : |x| = 1 \} \) and \( \mu(x) = \mu_{n-1}(x) \) is the normalized invariant (Haar) measure on \( S^{n-1} \). Let \( \|x\|^* = \sup_{y \neq 0} ((x, y) \|y\|) \). Then \( (1/b)|x| \leq \|x\|^* \leq a|x| \), and we define \( M_{r,*} = \int_{S^{n-1}} \|x\|^* \, d\mu(x) \).

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Let $K = \{ x \in X : \|x\| \leq 1 \}$, $K^* = \{ x \in X : \|x\|^* \leq 1 \}$, and $D = \{ x \in X : |x| \leq 1 \}$. We consider also the usual ($n$-dimensional) Lebesgue measure $(\text{Vol}_n)$ on $\mathbb{R}^n$ normalized (for example) so that the induced measure on $S^{n-1}$ coincides with $\mu(x)$; that is,

$$\text{Vol}_n(D) = (1/n)\mu(S^{n-1}) = 1/n.$$

We will use the following geometrical inequalities:

1. $(\text{Vol}_n K / \text{Vol}_n D)^{1/n} \leq M_{\star^*}$ (the Urysohn inequality [U]),
2. $\text{Vol}_n K \cdot \text{Vol}_n K^* \leq (\text{Vol}_n D)^2$ (the Santalo inequality [S]).

Also define $M_{\star^*} \cdot M_{\star^*} = M$.

3. We prove the following proposition (see also [M2]).

**Proposition.** For every $\lambda$, $0 < \lambda < 1$, there exists a subspace $E \subset X$, $\text{dim } E \geq \lambda n$, such that $E^*$ contains a subspace $F \subset E^*$, $\text{dim } F = k \geq \lambda^2 n$, such that

$$d(F, l^*_1) \leq [C_1(M + 1)]^{1/(1 - \lambda^2)},$$

where $C_1$ is an absolute constant (say $\sim 8\pi$).

**Proof.** We start with a general argument valid for an arbitrary euclidean norm $| \cdot |$ on $\mathbb{R}^n$. This norm will be defined in §§5 and 6. We introduce an additional norm $\| \cdot \|_1$ on $X$ such that

$$K_1 = \{ x \in X : \|x\|_1 \leq 1 \} = \text{Conv}(K \cup D).$$

Then $K_1^* = \{ x \in X : \|x\|^*_1 \leq 1 \} = K^* \cap D$ (i.e., $\|x\|^*_1 = \max(\|x\|^*, |x|)$). Therefore,

$$M_{\star^*} \cdot \mu(x) \leq M_{\star^*} + 1.$$

Since $\|x\|_1 \leq |x|$, the so-called volume ratio of the pair $(K_1; D)$ is

$$\nu(K_1) = \frac{(\text{Vol}_n K_1 / \text{Vol}_n D)^{1/n} \leq M_{\star^*} + 1}{\text{def}} = A$$

(by (1)). Next we use the following statement, which is an immediate consequence of the technique of Szarek’s proof [Sz] of Kashin’s theorem [K] (for details see [M2]).

**Statement.** Let $\nu(K_1) < A$. Fix $0 < \lambda < 1$. Then for any $k \leq \lambda n$ there exists a subspace $E$, $\text{dim } E = k$, such that

$$\frac{1}{2}(2\pi A)^{-\theta} |x| \leq \|x\|_1 \leq \|x\|,$$

where $\theta = 1/(1 - \lambda)$. The normalized Haar measure $\nu_{n,k}$ of such subspaces in the Grassmann manifold $G_{n,k}$ of $k$-dimensional subspaces of $\mathbb{R}^n$ is at least $1 - 1/2^{n-1}$.

4. We return to the proof of Proposition 3 and consider the normalization $M_{\star} = 1$. Let $a = \{ x \in S : \|x\| \leq 2 \}$. Then obviously $\mu(a) \geq 1/2$. Fix $\delta > 0$. Let $B = \{ \xi \in G_{n,k} : \mu_\xi(a \cap \xi) \geq \delta \}$. Again, obviously, $\nu_{n,k}(B) = \gamma \geq (\frac{1}{2} - \delta)/(1 - \delta)$. Then for any $\xi \in B$,

$$\frac{\text{Vol}_\xi(K \cap \xi)}{\text{Vol}_\xi(D \cap \xi)} = \int_{S \cap \xi} \frac{1}{\|x\|^k} d\mu(x) \geq \frac{1}{2^k} \delta.$$

Choose $\delta = 1/4$ and so $\gamma \geq 1/3$. 

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Therefore, there exists a subspace $E_0$ (and actually a large measure of subspaces) as in Statement 4 such that

$$\operatorname{Vol}_k \left( K \cap E_0 \right) / \operatorname{Vol}_k D(E_0) \geq 1/2^k \cdot 1/4,$$

where $k = \dim E_0$. Consider now $E_0^*$. Then for any $x \in E_0^*$, $\|x\|_2 \leq (2\pi A)^{\theta} |x|$, and by (3) and the Santalo inequality (2),

$$\operatorname{Vol}_k \left( K \cap E_0 \right)^* / \operatorname{Vol}_k D(E_0) \leq 4 \cdot 2^k.$$

Therefore, introducing a new norm on $E_0$: $\|x\|_2 = \|x\|_* / (2\pi A)^{\theta}$, we have that $K_2 = \{ x : \|x\|_2 \leq 1 \}$ has the volume ratio

$$\nu_r(K_2) \leq A_1 = 4(2\pi A)^{\theta} \cdot 4^{1/k} \quad (\text{and } \|x\|_2 \leq |x|).$$

So we may use Statement 4 one more time for the norm $\| \cdot \|_2$ to finish the proof of Proposition 3.

6. Proposition 3 contains a number $M$ which depends on the choice of a euclidean norm in $R^n$. It is known [F, T.] that for every $(X, \| \cdot \|)$ there exists a euclidean norm $| \cdot |$ such that $M + 1 \leq c_2 \| \text{Rad}_X \|$, where $c_2$ is an absolute constant and $\| \text{Rad}_X \|$ is the norm of the so-called Rademacher projection of $L_2(X)$ onto Rad $X$, which, as G. Pisier [P] has proved, may be estimated by

$$\| \text{Rad}_X \| \leq c_3 \ln (d_X + 1),$$

where $c_3$ is an absolute constant and $d_X = d(X, l^2_2) \leq \sqrt{n}$. Therefore, in particular,

$$\| \text{Rad}_X \| \leq c_3 \ln (n + 1).$$

7. Using 6, we may write in Proposition 3 that

$$d(F, l^2_2) \leq \left[ c \ln (d_X + 1) \right]^{1/(1 - \lambda)} \leq \left[ c \ln (n + 1) \right]^{1/(1 - \lambda)},$$

where $c$ is a universal constant. We now use Proposition 3 and (5) consecutively many times starting with $\lambda_1$, obtaining a space $F_1$ (as in Proposition 3), $\dim F_1 = k_1 \geq \lambda_1^2 n$ with

$$d_1 = d(F_1, l^2_2) \leq \left[ c \ln (n + 1) \right]^{1/(1 - \lambda)}.$$

For the second step we apply the same Proposition 3 to space $F_1$ (instead of $X$) with $\lambda_2$ and obtain a space $F_2$, $\dim F_2 = k_2 \geq (\lambda_2 \lambda_1)^2 n$ with

$$d_2 = d(F_2, l^2_2) \leq \left[ c \ln (d_1 + 1) \right]^{1/(1 - \lambda)}.$$

and so on.

It remains to state how we choose $\lambda_t$, $t = 1, 2, \ldots$. The notations $\ln^{(t)} A$ will be used for the $t$-times iterated logarithm of $A$ (so $\ln^{(2)} A = \ln \ln A$) if for any $k \leq t$, the $k$-iterated logarithm of $A$ is at least 2 and just 2 in the opposite case. With such an agreement we write, in (5), $\ln^{(1)} d_X (= \ln d_X)$ and $\ln^{(1)} n$ instead of $\ln (d_X + 1)$ and $\ln (n + 1)$.

For every $t \geq 1$, take $\lambda_t = 1 - \sqrt{6}/\ln^{(t+1)} n$, and we obtain, by using Proposition 3 $t$-times, a space $F_t$,

$$\dim F_t = k_t \geq \prod_{i=1}^t \left( 1 - \sqrt{6}/\ln^{(t+1)} n \right)^2 n.$$
and
\[ d_t = d(F_t, l_2^t) \leq (\ln d_{t-1})^{(\ln(t+1)n)^2/6}. \]
We assume now that \( c < \ln d_{t-1} \), and we just stop our iteration in the opposite case, \( d_{t-1} \leq e^c \). Therefore,
\[ d_t \leq (\ln d_{t-1})^{(\ln(t+1)n)^2/3} \]
and
\[ \ln d_t \leq \left( (\ln(t+1)n)^2/3 \right) \ln^{2} d_{t-1}. \]
Now,
\[ \ln d_1 \leq (\ln^2 n)^3/3 \leq (\ln^3 n)^3, \quad \ln d_2 \leq (\ln^3 n)^3, \]
and, in general,
\[ \ln d_r \leq (\ln^3 n)^3. \]
Now take \( d \) from the statement of Theorem 1. By Remark 1 we may assume \( d \) large enough; so let \( \ln d > e^c \) and \( a = (\ln \ln d)/3 > \sqrt{6} \cdot 2 \). We stop our iteration procedure for \( r \) such that for the last time, \( \ln(t+1)n \geq a \) (i.e., \( e^a > \ln(t+1)n \)), which implies, by (8) and (7), \( d_r \leq \exp(e^{3a}) = d \). Of course, the iteration could have stopped before because of the first condition if, for some \( j < t \), \( d_j \leq e^c < d \). Therefore, in both cases we have found a space \( F_j, j \leq t \), such that \( d(F_j, l_2^j) \leq d \). It remains to estimate \( \text{dim} F_t = k_t \), using (6). On the last step of the iteration we have \( \ln(t+1)n \geq a \) (\( > 2\sqrt{6} \)), and, therefore, \( \lambda_t \geq (1 - \sqrt{6}/a), \lambda_{t-1} \geq (1 - \sqrt{6}/e^a) \), and so on; so it is enough to estimate from below the infinite product
\[ (1 - \sqrt{6}/a) \cdot (1 - \sqrt{6}/e^a) \cdot \cdots \cdot (1 - \sqrt{6}/e^b) \cdots \geq \prod_{t=0}^{\infty} (1 - \sqrt{6}/a^{2^t}) = f(a) \to 1 \]
if \( a \to \infty \), and, therefore, we prove the principal part of Theorem 1. It is also clear that the main part of this product is the first term \( f(a) \approx 1 - \sqrt{6}/a = 1 - 3\sqrt{6}/\ln \ln d \).

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Added in Proof. I have now obtained the following better estimate for the function \( \lambda(d) \) in Theorem 1:
\[ \lambda(d) \geq 1 - c \sqrt{\frac{\log d}{d}} \quad \text{for large } d. \]

References


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