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ALMOST EUCLIDEAN QUOTIENT SPACES OF SUBSPACES OF A FINITE-DIMENSIONAL NORMED SPACE

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ABSTRACT. The main result of this article is Theorem 1 which states that a quotient space Y, $\dim Y = k$, of a subspace of any finite dimensional normed space X, $\dim X = n$, may be chosen to be d-isomorphic to a euclidean space even for $k = [\lambda n]$ for any fixed $\lambda < 1$ (and d depending on λ only).

The following theorem is proved.

1. Theorem. *For every $d > 1$ there exists $\lambda(d) > 0$ such that every n-dimensional normed space X contains a k-dimensional quotient space F of a subspace $E \subset X$ which satisfies*

(i) $$d(F, l_2^k) \leqslant d,$$

(ii) $$\dim F = k \geqslant \lambda(d)n.$$

(*Here $d(F, l_2^k)$ denotes a Banach-Mazur distance between two normed spaces*; *i.e.*,

$$d(X, Y) = \inf\{\|T\| \cdot \|T^{-1}\| \text{ over all linear isomorphisms } T\colon X \to Y\}.)$$

Moreover, $\lambda(d) \to 1$ *if* $d \to \infty$ *and, for large* d, $\lambda(d) \simeq 1 - 3\sqrt{6}/\ln \ln d$.

REMARK 1. It is enough to prove Theorem 1 for large d only, because, as proved in [$\mathbf{M}_1$], any d-isomorphic copy of l_2^m contains, for any $\varepsilon > 0$, a $(1 + \varepsilon)$-isomorphic copy of l_2^k, where $k \geqslant \kappa(\varepsilon)m/d^2$ and $\kappa(\varepsilon) > 0$ depends on $\varepsilon > 0$ only.

REMARK 2. Of course, the theorem states that the dual E^* to $E \subset X$ contains a subspace $F^* \subset E^*$ which satisfies (i) and (ii) of the theorem.

REMARK 3. In [$\mathbf{M}_2$] the theorem was proved with a logarithmic factor, and this theorem was formulated as a problem. We refer the reader to this paper for relevant discussion.

2. Notations. Let X be an n-dimensional normed space, i.e., R^n with the norm $\|\cdot\|$, and let (x, y) be an inner product on X; consequently, $|x| = (x, x)^{1/2}$ is a euclidean norm on X. For any $x \in X$ let $(1/a)|x| \leqslant \|x\| \leqslant b|x|$ and $M_r = \int_{x \in S^{n-1}} \|x\|\, d\mu(x)$, where $S^{n-1} = \{x \in X\colon |x| = 1\}$ and $\mu(x) \equiv \mu_{n-1}(x)$ is the normalized invariant (Haar) measure on S^{n-1}. Let $\|x\|^* = \sup_{y \neq 0}(|(x, y)|/\|y\|)$. Then $(1/b)|x| \leqslant \|x\|^* \leqslant a|x|$, and we define $M_{r*} = \int_{S^{n-1}} \|x\|^*\, d\mu(x)$.

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Let $K = \{x \in X\colon \|x\| \leqslant 1\}$, $K^* = \{x \in X\colon \|x\|^* \leqslant 1\}$, and $D = \{x \in X\colon |x| \leqslant 1\}$. We consider also the usual (n-dimensional) Lebesgue measure (Vol_n) on R^n normalized (for example) so that the induced measure on S^{n-1} coincides with $\mu(x)$; that is,

$$\mathrm{Vol}_n(D) = (1/n)\mu(S^{n-1}) = 1/n.$$

We will use the following geometrical inequalities:

$$(1) \qquad (\mathrm{Vol}_n K/\mathrm{Vol}_n D)^{1/n} \leqslant M_{r^*} \quad \text{(the Urysohn inequality [U])},$$

and

$$(2) \qquad \mathrm{Vol}_n K \cdot \mathrm{Vol}_n K^* \leqslant (\mathrm{Vol}_n D)^2 \quad \text{(the Santalo inequality [S])}.$$

Also define $M_r \cdot M_{r^*} = M$.

3. We prove the following proposition (see also [**M**$_2$]).

PROPOSITION. *For every* λ, $0 < \lambda < 1$, *there exists a subspace* $E \subset X$, $\dim E \geqslant \lambda n$, *such that* E^* *contains a subspace* $F \subset E^*$, $\dim F = k \geqslant \lambda^2 n$, *such that*

$$d(F, l_2^k) \leqslant [C_1(M+1)]^{1/(1-\lambda)^2},$$

where C_1 *is an absolute constant (say* $\sim 8\pi$).

PROOF. We start with a general argument valid for an arbitrary euclidean norm $|\cdot|$ on R^n. This norm will be defined in §§5 and 6. We introduce an additional norm $\|\cdot\|_1$ on X such that

$$K_1 = \{x \in X\colon \|x\|_1 \leqslant 1\} = \mathrm{Conv}(K \cup D).$$

Then $K_1^* = \{x \in X\colon \|x\|_1^* \leqslant 1\} = K^* \cap D$ (i.e., $\|x\|_1^* = \max(\|x\|^*, |x|)$). Therefore,

$$M_{r_1^*} = \int_{S^{n-1}} \|x\|_1^* \, d\mu(x) \leqslant M_{r^*} + 1.$$

Since $\|x\|_1 \leqslant |x|$, the so-called volume ratio of the pair $(K_1; D)$ is

$$\mathrm{vr}(K_1) \stackrel{\mathrm{def}}{=} (\mathrm{Vol}_n K_1/\mathrm{Vol}_n D)^{1/n} \leqslant M_{r^*} + 1 \stackrel{\mathrm{def}}{=} A$$

(by (1)). Next we use the following statement, which is an immediate consequence of the technique of Szarek's proof [**Sz**] of Kashin's theorem [**K**] (for details see [**M**$_2$]).

4. Statement. Let $\mathrm{vr}(K_1) \leqslant A$. Fix $0 < \lambda < 1$. Then for any $k \leqslant \lambda n$ there exists a subspace E, $\dim E = k$, such that

$$\tfrac{1}{2}(2\pi A)^{-\theta}|x| \leqslant \|x\|_1 \leqslant \|x\|,$$

where $\theta = 1/(1-\lambda)$. The normalized Haar measure $\nu_{n,k}$ of such subspaces in the Grassmann manifold $G_{n,k}$ of k-dimensional subspaces of R^n is at least $1 - 1/2^{n-1}$.

5. We return to the proof of Proposition 3 and consider the normalization $M_r = 1$. Let $a = \{x \in S\colon \|x\| \leqslant 2\}$. Then obviously $\mu(a) \geqslant 1/2$. Fix $\delta > 0$. Let $B = \{\xi \in G_{n,k}\colon \mu_\xi(a \cap \xi) \geqslant \delta\}$. Again, obviously, $\nu_{n,k}(B) = \gamma \geqslant (\frac{1}{2} - \delta)/(1-\delta)$. Then for any $\xi \in B$,

$$\frac{\mathrm{Vol}_\xi(K \cap \xi)}{\mathrm{Vol}_\xi(D \cap \xi)} = \int_{S \cap \xi} \frac{1}{\|x\|^k} d\mu(x) \geqslant \frac{1}{2^k}\delta.$$

Choose $\delta = 1/4$ and so $\gamma \geqslant 1/3$.

Therefore, there exists a subspace E_0 (and actually a large measure of subspaces) as in Statement 4 such that

(3) $$\mathrm{Vol}_k(K \cap E_0)/\mathrm{Vol}_k D(E_0) \geqslant 1/2^k \cdot 1/4,$$

where $k = \dim E_0$. Consider now E_0^*. Then for any $x \in E_0^*$, $\|x\|^* \leqslant 2(2\pi A)^\theta |x|$, and by, (3) and the Santalo inequality (2),

$$\mathrm{Vol}_k(K \cap E_0)^*/\mathrm{Vol}_k D(E_0) \leqslant 4 \cdot 2^k.$$

Therefore, introducing a new norm on E_0: $\|x\|_2 = \|x\|^*/2(2\pi A)^\theta$, we have that $K_2 = \{x\colon \|x\|_2 \leqslant 1\}$ has the volume ratio

$$\mathrm{vr}(K_2) \leqslant A_1 = 4(2\pi A)^\theta \cdot 4^{1/k} \quad (\text{and } \|x\|_2 \leqslant |x|).$$

So we may use Statement 4 one more time for the norm $\|\cdot\|_2$ to finish the proof of Proposition 3.

6. Proposition 3 contains a number M which depends on the choice of a euclidean norm in R^n. It is known **[F. T.]** that for every $(X, \|\cdot\|)$ there exists a euclidean norm $|\cdot|$ such that $M + 1 \leqslant c_2 \|\mathrm{Rad}_X\|$, where c_2 is an absolute constant and $\|\mathrm{Rad}_X\|$ is the norm of the so-called Rademacher projection of $L_2(X)$ onto Rad X, which, as G. Pisier **[P]** has proved, may be estimated by

(4) $$\|\mathrm{Rad}_X\| \leqslant c_3 \ln(d_X + 1),$$

where c_3 is an absolute constant and $d_X = d(X, l_2^n) \leqslant \sqrt{n}$. Therefore, in particular,

$$\|\mathrm{Rad}_X\| \leqslant c_3 \ln(n+1).$$

7. Using 6, we may write in Proposition 3 that

(5) $$d(F, l_2^k) \leqslant [c \ln(d_X + 1)]^{1/(1-\lambda)^2} \leqslant [c \ln(n+1)]^{1/(1-\lambda)^2},$$

where c is a universal constant. We now use Proposition 3 and (5) consecutively many times starting with λ_1, obtaining a space F_1 (as in Proposition 3), $\dim F_1 = k_1 \geqslant \lambda_1^2 n$ with

$$d_1 = d(F_1, l_2^{k_1}) \leqslant [c \ln(n+1)]^{1/(1-\lambda_1)^2}.$$

For the second step we apply the same Proposition 3 to space F_1 (instead of X) with λ_2 and obtain a space F_2, $\dim F_2 = k_2 \geqslant (\lambda_2\lambda_1)^2 n$ with

$$d_2 = d(F_2, l_2^{k_2}) \leqslant [c \ln(d_1 + 1)]^{1/(1-\lambda_2)^2},$$

and so on.

It remains to state how we choose λ_t, $t = 1, 2, \ldots$. The notations $\ln^{(t)} A$ will be used for the t-times iterated logarithm of A (so $\ln^{(2)} A = \ln\ln A$) if for any $k \leqslant t$, the k-iterated logarithm of A is at least 2 and just 2 in the opposite case. With such an agreement we write, in (5), $\ln^{(1)} d_X$ $(= \ln d_X)$ and $\ln^{(1)} n$ instead of $\ln(d_X + 1)$ and $\ln(n+1)$.

For every $t \geqslant 1$, take $\lambda_t = 1 - \sqrt{6}/\ln^{(t+1)} n$, and we obtain, by using Proposition 3 t-times, a space F_t,

(6) $$\dim F_t = k_t \geqslant \prod_{i=1}^{t} \left(1 - \sqrt{6}/\ln^{(i+1)} n\right)^2 n$$

and

$$d_t = d\left(F_t, l_2^{k_t}\right) \leqslant (c \ln d_{t-1})^{(\ln^{(t+1)} n)^2/6}.$$

We assume now that $c < \ln d_{t-1}$, and we just stop our iteration in the opposite case, $d_{t-1} \leqslant e^c$. Therefore,

$$d_t \leqslant (\ln d_{t-1})^{(\ln^{(t+1)} n)^2/3}$$

and

$$\ln d_t \leqslant \left((\ln^{(t+1)} n)^2/3\right) \ln^{(2)} d_{t-1}. \tag{7}$$

Now,

$$\ln d_1 \leqslant (\ln^{(2)} n)^3/3 \leqslant (\ln^{(2)} n)^3, \quad \ln d_2 \leqslant (\ln^{(3)} n)^3,$$

and, in general,

$$\ln d_t \leqslant (\ln^{(t+1)} n)^3. \tag{8}$$

Now take d from the statement of Theorem 1. By Remark 1 we may assume d large enough; so let $d > e^c$ and $a = (\ln \ln d)/3 > \sqrt{6} \cdot 2$. We stop our iteration procedure for t such that for the last time, $\ln^{(t+1)} n \geqslant a$ (i.e., $e^a > \ln^{(t+1)} n$), which implies, by (8) and (7), $d_t \leqslant \exp(e^{3a}) = d$. Of course, the iteration could have stopped before because of the first condition if, for some $j < t$, $d_j \leqslant e^c < d$. Therefore, in both cases we have found a space $F_j, j \leqslant t$, such that $d(F_j, l_2^{k_j}) \leqslant d$. It remains to estimate $\dim F_t = k_t$ using (6). On the last step of the iteration we have $\ln^{(t+1)} n \geqslant a\ (> 2\sqrt{6})$, and, therefore, $\lambda_t \geqslant (1 - \sqrt{6}/a)$, $\lambda_{t-1} \geqslant (1 - \sqrt{6}/e^a)$, and so on; so it is enough to estimate from below the infinite product

$$(1 - \sqrt{6}/a) \cdot (1 - \sqrt{6}/e^a) \cdots \left(1 - \sqrt{6}/e^{e^{\cdot^{\cdot^{e^a}}}}\right) \cdots \geqslant \prod_{t=0}^{\infty} \left(1 - \sqrt{6}/a^{2^t}\right) = f(a) \to 1$$

if $a \to \infty$, and, therefore, we prove the principal part of Theorem 1. It is also clear that the main part of this product is the first term $f(a) \simeq 1 - \sqrt{6}/a = 1 - 3\sqrt{6}/\ln \ln d$.

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ADDED IN PROOF. I have now obtained the following better estimate for the function $\lambda(d)$ in Theorem 1:

$$\lambda(d) \geqslant 1 - c\sqrt{\frac{\log d}{d}} \quad \text{for large } d.$$

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