ANALYTICITY IN THE BOUNDARY OF A PSEUDOCONVEX DOMAIN

ALAN V. NOELL

Abstract. Let $D$ be a bounded pseudoconvex domain with $C^\infty$ boundary in $\mathbb{C}^n$, $A^\infty(D)$ the algebra of functions holomorphic in $D$ and $C^\infty$ up to the boundary, and $M$ a compact real-analytic manifold in the boundary which is integral for the complex structure of the boundary and which has no complex tangent vectors. A necessary and sufficient condition that each element of $A^\infty(D)$ be real-analytic on $M$ is that the germ of the complexification of $M$ be in the boundary. Examples indicate that the quasi-analyticity of $A^\infty(D)$ along $M$ is possible even in the absence of complex manifolds in the boundary.

1. Introduction. We call a smooth manifold $M$ in the boundary of a domain an integral manifold if its tangent space at each point is contained in the maximal complex subspace of the tangent space of the boundary. $M$ is totally real if it has no complex tangent vectors; more precisely, if $J$ is the almost complex structure, the condition is that $T_p(M) \cap JT_p(M) = 0$ for all $p \in M$. A well-known theorem due to Stein states that holomorphic functions which are Lipschitz on $D$ are twice as smooth when restricted to integral curves. (For the precise statement we refer the reader to [9, Corollary 2, p. 443].) In this note we investigate what conditions on $D$ (or $\partial D$) imply high regularity of functions in $A^\infty(D)|M = \{ f|_M; f \in A^\infty(D) \}$; here $M$ is a compact totally real real-analytic integral manifold in $\partial D$. Our results depend on the notion of a complexification of such a manifold. Suppose $M$ has real dimension $m$. Locally (near $p \in M$) we take a real-analytic parametrization $\phi: V \to M$, where $V$ is a neighborhood of 0 in $\mathbb{R}^m$ and $\phi(0) = p$. The holomorphic extension $\Phi$ of $\phi$ to a neighborhood $V'$ of 0 in $\mathbb{C}^m$ is nonsingular since $M$ is totally real; then $\Phi(V')$ is a complexification of $M$ near $p$. Using the compactness of $M$ we combine these to get a complex submanifold $M'$ of a neighborhood $W$ of $M$ which has complex dimension $m$ and which contains $M$ as a submanifold. Details of this construction are in [10, p. 1274]. Note that, assuming the connectedness of $M' \cap W$, for each real-analytic function on $M$ there are a neighborhood $W'$ of $M$ and a unique extension of the function to $H(W' \cap M')$. (Here, as elsewhere, $H(N)$ denotes the algebra of holomorphic functions on the (connected) complex manifold $N$.) Our main result can then be stated as follows.

Theorem. Let $D$ be a bounded pseudoconvex domain in $\mathbb{C}^n$ with $C^\infty$ boundary, $M$ a compact totally real real-analytic integral manifold in $\partial D$, and $M'$ a complexification of $M$ in $W$. Then each element of $A^\infty(D)|M$ is real-analytic if and only if there is a neighborhood $U \subseteq W$ of $M$ so that $U \cap M' \subseteq \partial D$.
The proof of this theorem is in §2. We remark that obviously pseudoconvexity is required in the theorem; furthermore, some minimal smoothness of the boundary is necessary. In fact, Sibony constructed in [8, p. 973] a bounded pseudoconvex domain in \( C^2 \) (with nonsmooth boundary) so that all bounded holomorphic functions on the domain extend to be holomorphic on a strictly larger domain.

Motivation for this work came from a study of interpolation in [6]; there an example is given of a class of domains for which \( A^\infty(D) \) gains a good deal of smoothness upon restriction to an integral curve. In §3 we further discuss this example as a contrast to the theorem above. In particular, we give the following

**Example.** There exists a convex domain \( D \subseteq C^2 \) which is strongly pseudoconvex off of a line segment \( K \) so that \( A^\infty(D) \) is quasi-analytic along a subinterval of \( K \).

Our proof of the theorem depends on the identification of the spectrum of the algebra \( A^\infty \) given by Hakim and Sibony in [3, Theorem 1, p. 128]. Recall that \( A^\infty(D) \) is a Fréchet algebra with the family of norms given by

\[
P_N(f) = \sum_{\left| \alpha \right| \leq N} \frac{1}{\alpha!} \left\| D^\alpha f \right\|_D;
\]

here, as elsewhere in this note, \( \left\| g \right\|_X \) denotes the supremum of \( |g| \) on \( X \).

**Theorem (Hakim-Sibony).** If \( D \) is a bounded pseudoconvex domain with \( C^\infty \) boundary, then the space of nonzero continuous complex homomorphisms of \( A^\infty(D) \) can be identified with \( \bar{D} \).

2. **Proof of the theorem.** Suppose that, for some neighborhood \( U \) of \( M \), \( U \cap M' \subseteq \partial D \). If \( f \in A^\infty(D) \), then \( \bar{\partial} f \equiv 0 \) in \( \bar{D} \), so \( f \) is holomorphic on \( U \cap M' \). It follows that \( f \) is real-analytic on \( M \). Thus each element of \( A^\infty(D)|M \) is real-analytic.

For the nontrivial part of the proof, we assume each element of \( A^\infty(D)|M \) is real-analytic and fix a point \( p \in M \). For each \( f \in A^\infty(D)|M \) there is a neighborhood \( V \) of \( p \) (depending on \( f \)) so that \( f \) extends to be holomorphic on \( V \cap M' \). Our first step is to remove the apparent dependence of \( V \) on \( f \) (cf. the argument in [3, p. 131]). Let \( B(r) \) denote the open ball with center \( p \) and radius \( r > 0 \); let \( X(r) \) be the Fréchet space of pairs \((F,f)\) with \( F \in H(B(r) \cap M'), f \in A^\infty(D) \), and \( F = f \) on \( B(r) \cap M \); and, let \( \rho(r) : X(r) \to A^\infty(D) \) be the restriction map. We know that the union of the images of \( \rho(r) \) over \( 1/r = 1,2,3, \ldots \) is \( A^\infty(D) \), so, for some \( r_1 \), the image of \( \rho(r_1) \) is of the second category in \( A^\infty(D) \). By the open mapping theorem for Fréchet spaces (e.g., [7, p. 47]), \( \rho(r_1) \) is surjective. Thus, if \( V = B(r_1) \), each element of \( A^\infty(D)|M \) extends to be holomorphic on \( V \cap M' \).

The second step is to show that \( V \cap M' \subseteq \bar{D} \). Fix a point \( q \in V \cap M' \) and define a complex homomorphism \( \chi : A^\infty(D) \to C \) by \( \chi(f) := F(q) \) if \( f \in A^\infty(D) \) and \( F \) is an extension of \( f \) which is holomorphic on \( V \cap M' \). Since the extension is unique, \( \chi \) is well defined, and the following argument shows that \( \chi \) is continuous: If \( g \in A^\infty(D) \), then \( |\chi(g)| \leq \|g\|_{C\bar{D}} \), for otherwise \( g - \chi(g) \) would be invertible in \( A^\infty(D) \), an impossibility. Thus, if \( f_j \to f \) in \( A^\infty(D) \), from \( \|f_j - f\|_D \to 0 \) it follows that \( \chi(f_j) \to \chi(f) \). Hence, \( \chi \) is continuous. By the aforementioned result of Hakim and Sibony, \( \chi \) is given by evaluation at a point of \( \bar{D} \), and it is clear that this point must be \( q \). If follows that \( q \in \bar{D} \) and so \( V \cap M' \subseteq \bar{D} \).
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The third step is to show that, in fact, \( V \cap M' \subseteq \partial D \). For this we use the fact that there is a function \( \sigma \in C(\bar{D}) \) which is plurisubharmonic on \( D \) and satisfies \( \sigma < 0 \) on \( D \) while \( \sigma = 0 \) on \( \partial D \); this is a simple form of the theorem of Diederich and Fornaess [2, Theorem 1, p. 131] on bounded plurisubharmonic exhaustion functions. We claim that \( \sigma \) is actually plurisubharmonic on \( V \cap M' \). To see this, fix \( q \in V \cap M' \), let \( n \) be the outward unit normal to \( \partial D \) at \( q \), and let \( V' \subset V \) be a small neighborhood of \( q \). Since \( V \cap M' \subseteq \bar{D} \), if \( \varepsilon > 0 \) is small, then
\[
\{ t - \varepsilon n; \ t \in V' \cap M' \} \subseteq D.
\]
Thus \( \sigma(t) \) is the uniform limit on \( V' \cap M' \) of the plurisubharmonic functions \( \sigma_\varepsilon(t) := \sigma(t - \varepsilon n) \) as \( \varepsilon \to 0 \); it follows that \( \sigma \) is plurisubharmonic on \( V' \cap M' \). Since \( q \) was arbitrary, \( \sigma \) is plurisubharmonic on \( V \cap M' \), giving the claim. Now \( \sigma \) attains its maximum value at the (relative) interior point \( p \) of \( V \cap M' \); by the maximum principle, \( \sigma \equiv 0 \) on \( V \cap M' \). Thus \( V \cap M' \subseteq \partial D \).

We have shown that, for each \( p \in M \), there exists a neighborhood \( V \) of \( p \) so that \( V \cap M' \subseteq \partial D \). It follows that there is a neighborhood \( U \subseteq W \) of \( M \) so that \( U \cap M' \subseteq \partial D \).

REMARK. If \( A(D) := H(D) \cap C(\bar{D}) \), then it is easy to see that the assumption that \( U \cap M' \subseteq \partial D \) for a neighborhood \( U \) of \( M \) implies that each element of \( A(D)|M \) is real-analytic. In fact, fixing \( f \in A(D) \) and \( q \in U \cap M' \), we get that \( f \) is locally near \( q \) the uniform limit on \( M' \) of holomorphic functions by arguing as for \( \sigma \) in step 3 above. It follows that \( f|M \) is real-analytic.

3. Example of quasi-analyticity in the boundary. For the example we choose two nonnegative even functions \( \phi \) and \( \chi \) in \( C^\infty(\mathbb{R}) \) so that
(a) each is strictly convex off its zero set;
(b) \( \chi^{-1}(0) = [-2, 2] \);
(c) \( \phi^{-1}(0) = \{0\} \); and
(d) \( \phi \) vanishes to infinite order at \( 0 \).

From [6, Example 4.1] we recall the domain \( D \), defined near \( K := [-2, 2] \times \{0\} \) in \( \mathbb{C}^2 \), by
\[
D := \left\{ (z, w); u + \chi(x) + \phi(y) + v^2 \left( 1 + \frac{1}{100} |z|^2 \right) < 0 \right\};
\]
here we use the notation \( z = x + iy, \ w = u + iv \). \( D \) is convex and strongly pseudoconvex off of \( K \), and \( K \) is an integral curve. We put \( L := [-1, 1] \times \{0\} \) and
\[
I_k = I_k(\phi) := \int_0^1 \phi'(t) t^{-k} dt \quad \text{for } k \geq 1.
\]

LEMMA 1. Given \( f \in A^\infty(D) \) there exists \( C > 0 \) so that
\[
\| \partial^k f / \partial x^k \|_L \leq C k! I_k \quad \text{for } k \geq 1.
\]

PROOF. Lemma 4.1 of [6] gives this estimate with \( L \) replaced by \( \{(0,0)\} \), and one only needs to check that the estimate holds uniformly on \( L \). For the convenience of the reader, we sketch the proof. If \( k \geq 1 \) then
\[
\frac{\partial^k f}{\partial x^k}(a, 0) = -\int_0^1 \frac{d}{dt} \left[ \frac{\partial^k f}{\partial x^k}(a, -\phi(t)) \right] dt + \frac{\partial^k f}{\partial x^k}(a, -\phi(1))
\]
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whenever $-1 \leq a \leq 1$. The integrand is bounded above by $k!\|\partial f/\partial w\|_{D} \phi'(t)t^{-k}$ because of the Cauchy estimates for $\partial f/\partial w$ on discs in $D$ of the form 
\[
\{z; \ |z - a| \leq t\} \times \{-\phi(t)\};
\]
the second term is similarly bounded above by $k!\|f\|_{D}$. This gives the desired estimate.

The lemma shows that we can get good regularity for $A^{\infty}(D)|_{L}$ by choosing $\phi$ so that $I_{k}(\phi)$ grows slowly with $k$. The proof of the main theorem shows that we cannot choose $\phi$ so that, for some $C_{1} > 0$,
\[
(*) \quad I_{k}(\phi) \leq C_{1}^{k} \quad \text{for } k \geq 1.
\]
Here is a more direct proof of this: Put
\[
\psi(t) := \begin{cases} 
0 & \text{if } t < 1, \\
\phi'(1/t) & \text{if } t \geq 1.
\end{cases}
\]
The holomorphic Fourier transform $F$ of $\psi$ defined by
\[
F(z) := \int_{-\infty}^{\infty} \psi(t)e^{itz} dt \quad (z \in \mathbb{C})
\]
would, if (*) held, be an entire function of exponential type (a simple estimate); by the Paley-Wiener Theorem, $F$ would be the Fourier transform of a function with compact support, so $\psi$ would have compact support. Thus (*) implies $\phi \equiv 0$ near 0, contradicting (c) above. In the following lemma we indicate one possible construction of a $\phi$ whose growth rate approximates (*).

**Lemma 2.** Suppose $\{a_{k}\}$ is an unbounded increasing sequence with $a_{1} \geq 1$. Then there exists a function $\phi$ of the required form with
\[
I_{k}(\phi) \leq a_{k}^{k} \quad \text{for } k \geq 1.
\]

**Proof.** Fix $\lambda \in C^{\infty}(\mathbb{R})$ so that $0 \leq \lambda \leq 1, \lambda(t) \equiv 0$ if $t \leq 1$, and $\lambda(t) \equiv 1$ if $t \geq 2$. If $j \geq 1$, let $c_{j} := \max\{|a_{k}\lambda^{(k)}(a_{j}t)|_{\mathbb{R}}; 0 \leq k \leq j\};$ then $1 \leq c_{j} < \infty$. We define
\[
\psi(t) := \sum_{j=1}^{\infty} \lambda(a_{j}t)t^{j}/(c_{j}j^{j}) \quad \text{for } t \geq 0.
\]
Then $\psi$ is infinitely differentiable, and $\psi > 0$ if $t > 0$. A rather crude estimate gives that, for $k \geq 2$,
\[
\int_{0}^{1} \psi(t)t^{-k} dt = \sum_{j=1}^{\infty} \int_{1/a_{j}}^{1} \lambda(a_{j}t)t^{j-k}/(c_{j}j^{j}) dt \\
\leq (k - 1)a_{k}^{k} + 1.
\]
If we choose $\phi$ to be even and satisfy $\phi(0) = \phi'(0) = 0$ while $\phi''(t) = \psi(t)$ for $t \geq 0$, then integration by parts gives that, for some $C_{1} > 0$, $I_{k}(\phi) \leq C_{1}a_{k}^{k}$ for $k \geq 1$. Dividing $\phi$ by $C_{1}$ gives the desired result.

**Example.** Let $a_{k} = \log k$ for $k \geq 3$, and let $\phi$ be the corresponding function given in Lemma 2. By Lemma 1, if $f \in A^{\infty}(D)$, then there exists $C > 0$ so that
\[
\|\partial^{k}f/\partial x^{k}\|_{L} \leq C(k \log k)^{k} \quad \text{for } k \geq 3.
\]
Since $\sum 1/(k \log k) = \infty$, the Denjoy-Carleman Theorem (e.g., [4, Chapter IV, pp. 101 ff.]) implies that $A^\infty(D)|L$ is quasi-analytic. We remark that with the choice $\chi(2 + t) = \phi(t)$ (for $t \geq 0$) it is straightforward to check that $A^\infty(D)|K$ is quasi-analytic.

The above example gives a result about peak sets for $A^\infty(D)$. Recall that a closed set $E$ in $\partial D$ is a peak set for $A^\infty(D)$ if there exists a function $g \in A^\infty(D)$ with $g = 0$ on $E$ while $\text{Re } g > 0$ on $D \setminus E$. $K$ is a peak set for $A^\infty(D)$ (take $g = -\omega$), but no subset $E$ of $(-1,1) \times \{0\}$ is a peak set for $A^\infty(D)$. In fact, if such a set $E$ were a peak set with corresponding function $g$, the function $f = \exp(-1/\sqrt{g}) \in A^\infty(D)$ would vanish to infinite order on $E$. By the quasi-analyticity of $A^\infty(D)|L$, $f \equiv 0$ on $L$, so $E \supseteq L$, a contradiction. (A different proof of a related fact about peak sets in $K$ is given in [5, Example 1.1].)

REMARK. In contrast to the above phenomena, $A(D)$ gains no regularity upon restriction to $K$ in the above examples. More precisely, $A(D)|K = C(K)$, i.e., $K$ is an interpolation set for $A(D)$. The proof is as follows: Since $K$ is a peak set, a well-known result from the theory of uniform algebras (e.g., [1, Corollary 2.4.3, p. 104]) implies that $A(D)|K$ is uniformly closed in $C(K)$. In addition, the Stone-Weierstrass Theorem implies that holomorphic polynomials are dense in $C(K)$. Thus $A(D)|K = C(K)$.

REFERENCES


DEPARTMENT OF MATHEMATICS 253-37, CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA, CALIFORNIA 91125