ON SURFACES IN R^4

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ABSTRACT. We provide answers (Theorem C) to some questions concerning surfaces in R^4 and maps into the quadric Q_2 raised by D. Hoffman and R. Osserman.

Let S be an oriented surface immersed in R^4. The Gauss map of S is the map G of S into G(2,4), the Grassmannian of oriented two-planes in R^4, given by G(p) = T_pS. G(2,4) can be identified with Q_2, the complex quadric in CP^3, and in turn Q_2 is biholomorphic to CP^1 \times CP^1. If we give CP^3 the Fubini-Study metric of constant holomorphic sectional curvature 2, then the induced metric on Q_2 is given by

\[ 2|dw_1|^2/(1 + |w_1|^2)^2 + 2|dw_2|^2/(1 + |w_2|^2)^2, \]

where (w_1, w_2) are coordinates on C \times C, viewed as local coordinates on CP^1 \times CP^1 [1]. The metric 2|dw|^2/(1 + |w|^2)^2 is the metric on C induced by the map of C onto S^2(1/√2) \subset R^3 given by w \mapsto σ^{-1}(√2 w), where σ^{-1} is inverse stereographic projection (with the sphere sitting on the xy-plane). Thus, Q_2 is isometric to S^2(1/√2) \times S^2(1/√2). In particular, if z is a local conformal parameter on S, then any map G of S into Q_2 splits into a pair of maps G(z) = (f_1(z), f_2(z)), where w_i = f_i(z) as above.

Now define the following quantities on S for i = 1, 2:

\[ F_i := \frac{f_i z}{1 + |f_i|^2}, \quad T_i(z) = \left[ \frac{(f_i z)^2}{(f_i z)} - \frac{2 \bar{f}_i f_{iz}}{1 + |f_i|^2} \right] \]

where \( f_{iz} \neq 0 \)

with the usual z and \( \bar{z} \) derivative notation. The following results are from [1, 2].

**THEOREM A.** For the Gauss map G of an oriented surface S immersed in R^4, write G = (f_1(z), f_2(z)) as above. Then we necessarily have

(1) \[ |F_1| = |F_2|, \]

and

(2) \[ \text{Im}\{ T_1 + T_2 \} = 0. \]

**THEOREM B.** Let S_0 be a simply connected Riemann surface (here and subsequently), let G = (f_1(z), f_2(z)) be some map of S_0 into Q_2, and define F_i and T_i as before, where z is a conformal parameter on S_0.

(i) If F_1 = F_2 = 0, then G is the Gauss map of a minimal surface in R^4, provided S_0 is not compact.
(ii) If \( F_1, F_2 \) are never zero, then \( G \) is the Gauss map of a surface \( S \) in \( \mathbb{R}^4 \) given by a conformal immersion of \( S_0 \) if and only if

(1) \[ |F_1| = |F_2|, \]

and

(2) \[ \text{Im}\{T_1 + T_2\} = 0. \]

Furthermore, in this case \( S \) is uniquely determined up to translation and homothety of \( \mathbb{R}^4 \).

Let \((1')\) denote the condition that \( F_1, F_2 \) are never zero (i.e., \( f_1^z \) and \( f_2^z \) are never zero) and \( |F_1| = |F_2| \). A special class of maps which satisfy (2) are harmonic maps, i.e., those \( f(z)'s \) such that

\[
L(f) := f_{zz} - 2\overline{f}f_z/(1 + |f|^2) = 0.
\]

In particular, if

(3) \[ L(f_i) = 0, \quad i = 1, 2, \]

then (2) is automatically satisfied. Condition (3) is simply that the map \( G: S_0 \to \mathbb{Q}_2 \) is harmonic. A theorem of Ruh and Vilms [4] asserts that the Gauss map of a submanifold of \( \mathbb{R}^n \) is harmonic if and only if the submanifold has parallel mean curvature. Combining this with Theorem B we now have the following observation:

A map \( G: S_0 \to \mathbb{Q}_2 \) is the Gauss map of a conformal immersion with parallel (nonzero) mean curvature in \( \mathbb{R}^4 \) if and only if \((1')\) and \( (3)\) hold.

Finally, an interesting subclass of surfaces of parallel mean curvature in \( \mathbb{R}^4 \) are minimal surfaces in some \( S^3(r) \). Hoffman and Osserman also prove the following

**Proposition.** A map \( G: S_0 \to \mathbb{Q}_2 \) is the Gauss map of a conformal minimal immersion of \( S_0 \) into some \( S^3(r) \) (viewed as sitting in \( \mathbb{R}^4 \)) if and only if \((1)\) and \( (3)\) are satisfied, as well as the following

(4) \[ f_{1z}/f_{1\overline{z}} = f_{2z}/f_{2\overline{z}}. \]

In view of these results, the following questions present themselves [1, 2]: Given a map from \( S_0 \) into \( S^2(1/\sqrt{2}) \), represented locally by \( f_1(z) \) as above, does there exist a map from \( S_0 \) into \( S^2(1/\sqrt{2}) \), represented by \( f_2(z) \), such that the pair \((f_1(z), f_2(z))\) satisfies

Q1. \((1')\) and \( (2)\)? Suppose \( f_1 \) satisfies \( L(f_1) = 0 \). Does there exist \( f_2(z) \) such that the pair \((f_1, f_2)\) satisfies

Q2. \((1')\) and \( (3)\), or

Q3. \((1'), (3)\) and \( (4)\)?

An affirmative answer to Q1 (Q2) would mean that the pair \((f_1, f_2)\) is the Gauss map of a conformal immersion (with parallel nonzero mean curvature) of \( S_0 \) in \( \mathbb{R}^4 \), while an affirmative answer to Q3 would mean that the pair \((f_1, f_2)\) is the Gauss map of a conformal minimal immersion of \( S_0 \) into some \( S^3(r) \) (viewed as sitting in \( \mathbb{R}^4 \)).
We answer Q2 and Q3 affirmatively in Theorem C. While this provides an affirmative answer to Q1 under the special assumption of (3) \((\Rightarrow (2))\), we do not know the answer to Q1 in general.

**Theorem C.** Given a map from \(S_0\), not conformally equivalent to \(S^2\), into \(S^2(1/\sqrt{2})\), written as \(f_1(z)\) as above, such that \(f_{1\bar{z}}\) is never zero, and \(L(f_1) = 0\), there exists a one-parameter family of maps of \(S_0\) into \(S^2(1/\sqrt{2})\), written as \(f_\theta(z)\), such that the pair \((f_1, f_\theta)\) satisfies (1) and (3). Furthermore, there is a unique \(\theta_0\) such that the pair \((f_1, f_{\theta_0})\) also satisfies (4). If \(S_0\) is conformally equivalent to \(S^2\), \(f_2 = f_1\) is the only possibility for even (1) and (3).

**Remark.** The idea of the proof is to regard \(f_1\) as the Gauss map of a surface \(S\) of constant (nonzero) mean curvature in \(\mathbb{R}^3\). The \(f_\theta\)'s are the Gauss maps of the associated family \(S_\theta, 0 \leq \theta \leq 2\pi, \) to \(S\). It turns out that condition (4) is then satisfied exactly for the surface \(S_\theta:\)

**Proof of Theorem C.** We regard \(f_1(z)\) as the representation of a map of \(S_0\) with \(S^2(1)\) as follows: Let \(\sigma(\sigma')\) be stereographic projection of \(S^2(1/\sqrt{2})\) (\(S^2(1)\)) onto \(\mathbb{C}\), and consider the transformation \(\mathbb{C} \rightarrow \mathbb{C}\) by

\[
\phi(w) = \frac{1}{2}(\sigma'(\sqrt{2} \sigma^{-1}(\sqrt{2} w))).
\]

\(\phi\) is just the identity map on \(\mathbb{C}\), so \(\phi(f_1) = f_1\). Replacing \(f_1\) by \(f_1 = \sqrt{2}(\sigma^{-1}(\sqrt{2} f_1)) \in S^2(1)\), and then representing \(f_1\) by \(\frac{1}{2}\sigma'(f_1)\), we see that we may regard \(f_1\) as a map into \(\mathbb{CP}^1\), with the metric \(4|dw|^2/(1 + |w|^2)^2\) of constant curvature 1. Thus it suffices to prove Theorem C with \(S^2(1/\sqrt{2})\) replaced by \(S^2(1)\). Now the conditions \(f_{1\bar{z}} \neq 0, L(f_1) = 0\) mean that \(f_1\) is a harmonic, nowhere anticonformal map of \(S_0\) into \(S^2 = S^2(1)\). From Hoffman-Osserman [1] and Kenmotsu [3], this guarantees that \(f_1\) is the Gauss map of a conformal immersion \(X\) of \(S_0\) into \(\mathbb{R}^3\) with constant nonzero mean curvature. If we specify that \(X(S_0)\) have constant mean curvature 1, then this determines \(S_\theta = X(S_\theta)\) up to translation in \(\mathbb{R}^3\). If \(S_0\) is conformally equivalent to \(S^2\), then \(S_\theta\) is the standard unit sphere, and any \(f_\theta\: S_0 \rightarrow S^2(1)\) satisfying (1) and (3) must come from the same \(X\) (up to translation of \(\mathbb{R}^3\)). For \(S_0\) not conformally \(S^2\), in the (global) isothermal parameter \(z\), the metric induced on \(S_\theta\) is given by

\[
4|f_{\theta \bar{z}}|^2/(1 + |\theta|^2)^2|dz|^2,
\]

and since \(S_\theta\) is isometric to \(S_\alpha\), we also have condition (1) satisfied for the pair \((f_0, f_\theta)\). Let \(\beta^\theta\) be the second fundamental form of \(S_\theta\). Then from formula 5.3 of [3], we have

\[
\frac{1}{F_\theta} \left\{ \frac{\beta^\theta_{11} - \beta^\theta_{12}}{2} - i\beta^\theta_{12} \right\} = \frac{f_{\theta \bar{z}}}{f_{\theta \bar{z}}},
\]

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From this we see that condition (4), in the presence of (1), is equivalent to

\[
\beta_{11}^0 - \beta_{22}^0 = \beta_{22}^\theta - \beta_{11}^\theta, \quad \beta_{12}^0 = -\beta_{12}^\theta.
\]

Finally, since

\[
\beta_{11}^\theta = \cos \theta (\beta_{11}^0 - F_0) + \sin \theta \beta_{12}^0 + F_0,
\]
\[
\beta_{22}^\theta = -\cos \theta (\beta_{11}^0 - F_0) - \sin \theta \beta_{12}^0 + F_0,
\]
\[
\beta_{12}^\theta = \cos \theta \beta_{12}^0 - \sin \theta (\beta_{11}^0 - F_0)
\]
and \(\beta_{11}^0 + \beta_{22}^0 = 2F_0\) [5], we see that condition (4) (cf. (6)) is equivalent to \(\theta = \pi\).

Q.E.D.

REFERENCES

2. _____, *The Gauss map of surfaces in \(\mathbb{R}^3\) and \(\mathbb{R}^4\)* (to appear).

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