A METRIC ON HYPERSPACES DEFINED BY WHITNEY MAPS

WŁODZIMIERZ J. CHARATONIK

Abstract. For a given continuum $X$ a new metric on the hyperspace $2^X$ is defined, which is equivalent to the Hausdorff distance, but which has some other properties.

All spaces in this paper are assumed to be metric and all mappings are continuous. A continuum is a compact connected space. Given a continuum $X$ with a metric $d$, we define the Hausdorff distance $H$ between two nonempty closed subsets $A$ and $B$ by

$$H(A, B) = \max\left\{ \sup_{a \in A} \inf_{b \in B} d(a, b), \sup_{b \in B} \inf_{a \in A} d(a, b) \right\}$$

(see [1, (0.4), p. 3]). The symbol $2^X$ denotes the hyperspace of all nonempty closed subsets of a continuum $X$ with the Vietoris topology (see [1, (0.11), p. 9] for the definition) or, equivalently (see [1, (0.13), p. 10]) with the topology determined by the Hausdorff distance.

A mapping $\mu : 2^X \to [0, \infty)$ is called a Whitney map (see [1, (0.50), p. 24]) if it satisfies the conditions:

1. for every $x \in X$, $\mu(\{x\}) = 0$; and
2. for every $A, B \in 2^X$ with $A \subset B$ and $A \neq B$, $\mu(A) < \mu(B)$.

We consider special Whitney maps, namely ones satisfying an additional condition:

3. for every $A, B \in 2^X$ with $A \subset B$ and for every $C \in 2^X$,

$$\mu(B \cup C) - \mu(A \cup C) \leq \mu(B) - \mu(A).$$

Such mappings do exist for every continuum $X$ (see Proposition 1 below).

Given a sequence of sets $\{A_n\}_{n=1}^\infty$ we denote by $\text{Ls} \ A_n$ the upper limit of the sequence in the sense of [1, (0.5), p. 4], and by $\text{Lim} \ A_n$ the limit of the sequence in the sense of [1, (0.5), p. 4] or, equivalently (see [1, (0.7), p. 4]), in the sense of the Hausdorff distance.

In the present paper a new metric on the hyperspace of a continuum is defined, which is equivalent to the Hausdorff distance, but which has some other properties.
We start with

**PROPOSITION 1.** For every continuum $X$ there are Whitney maps $\mu$ and $\mu'$ such that $\mu$ satisfies, while $\mu'$ does not satisfy, condition (3).

Really, the reader can verify that a Whitney map $\mu$ defined in [1, (0.50.2), p. 26] has property (3). On the other hand, let $x, y, z \in X$ be any distinct points and put $f((x)) = f((y)) = f((z)) = 0$, $f((x, y)) = f((x, z)) = f((y, z)) = 1$, and $f((x, y, z)) = 3$. Then $f$ satisfies (1) and (2) for the space $\{x, y, z\}$ and therefore it can be extended to a Whitney map $\mu'$ on $2^X$ (see [2, Corollary 3.4, p. 468] and observe that the assumption of connectedness of spaces is not used in the proof). However, putting $A = \{x\}, B = \{x, y\}$, and $C = \{z\}$, we can see that $f$ (and hence $\mu'$) does not satisfy (3).

**DEFINITION 2.** Let $X$ be a continuum and let $\mu$ be a Whitney map satisfying (3). Define, for every $P, Q \in 2^X$, 

$$D_\mu(P, Q) = \max\{\mu(P \cup Q) - \mu(P), \mu(P \cup Q) - \mu(Q)\}.$$  

**PROPOSITION 3.** $D_\mu$ defined above is a metric on $2^X$.

**PROOF.** The condition $D_\mu(P, Q) = 0$ if and only if $P = Q$ is a consequence of (2); the symmetry of $D_\mu$ is obvious from the definition. We show the triangle condition. Let $P, Q, R \in 2^X$. We can assume without loss of generality that $\mu(P) < \mu(Q)$. Then we have to show 

$$\mu(P \cup Q) - \min\{\mu(P), \mu(Q)\} + \mu(Q \cup R) - \min\{\mu(Q), \mu(R)\} \geq \mu(P \cup R) - \mu(P).$$

It is enough to show 

$$\mu(P \cup Q) - \mu(P) + \mu(Q \cup R) - \mu(Q) + \mu(P \cup R) + \mu(P) \geq 0,$$

but using (3) for $A = Q, B = P \cup Q$, and $C = R$ we see that the left member of the inequality is greater than or equal to 

$$\mu(P \cup Q \cup R) - \mu(Q \cup R) + \mu(Q \cup R) - \mu(P \cup R)$$

and, therefore, is nonnegative.

**PROPOSITION 4.** For any Whitney map $\mu$ satisfying (3) the metric $D_\mu$ is equivalent to the Hausdorff distance $H$.

**PROOF.** Let a set $A \in 2^X$ be given and assume a sequence $\{A_n\}_{n=1}^\infty$ tends to $A$ with respect to the Hausdorff distance, i.e., $H(A_n, A) \rightarrow 0$. Then $H(A_n \cup A, A) \rightarrow 0$, and by continuity of $\mu$ we have $\mu(A_n \cup A) \rightarrow \mu(A)$ and $\mu(A_n) \rightarrow \mu(A)$. Thus, 

$$\max\{\mu(A_n \cup A) - \mu(A), \mu(A_n \cup A) - \mu(A_n)\} \rightarrow 0,$$

i.e., the sequence $\{A_n\}_{n=1}^\infty$ tends to the set $A$ with respect to the metric $D_\mu$.

On the other hand assume $\{A_n\}_{n=1}^\infty$ tends to $A$ with respect to the metric $D_\mu$, i.e., 

(4) $\mu(A_n \cup A) - \mu(A) \rightarrow 0$ and

(5) $\mu(A_n \cup A) - \mu(A_n) \rightarrow 0$. 

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We show that

(6) \( \lim(\bigcup A_n) = A. \)

Assume, on the contrary, that there is a subsequence \( \{ A_n \}_{n=1}^\infty \) with \( \lim(\bigcup A_n) = B \neq A. \) Then \( A \subset B \) and (2) imply \( \mu(A) < \mu(B) \), a contradiction to (4).

Note that (6) implies

(7) \( Ls^c \subset A. \)

Now suppose there exists a subsequence \( \{ A_n \}_{n=1}^\infty \) with \( \lim A_n = C \neq A. \) By (7) we have \( C \subset A \) and, therefore, by (2), \( \mu(C) < \mu(A) \). Then (6) implies a contradiction to (5). So we have proved \( \lim A_n = A \), i.e., \( \{ A_n \}_{n=1}^\infty \) tends to \( A \) with respect to the Hausdorff distance.

Now we show some facts concerning the metric \( D_\mu \). Some of them are obvious and their proofs are omitted.

Let \( X \) be a fixed continuum and let \( \mu \) be a Whitney map satisfying (3).

**Fact 5.** Consider \( 2^X \) as a metric space with the metric \( D_\mu \), and let \( \mathcal{A} \subset 2^X \) be an ordered arc. Then \( \mu|_{\mathcal{A}}: \mathcal{A} \to [0, \infty) \) is an isometry.

**Fact 6.** Let \( x \in A \in 2^X. \) Then \( D_\mu(A, \{x\}) = \mu(A). \) In other words, the distance between a set and any point in the set does not depend on the choice of the point.

**Fact 7.** Let \( \mathcal{A} \) be an ordered arc contained in \( 2^X \) and let \( P \in 2^X. \) Denote by \( A_0 \) either the only set in \( \mathcal{A} \) satisfying \( \mu(A_0) = \mu(P) \) if such a set does exist, or \( \bigcap \mathcal{A} \) if \( \mu(P) < \mu(A) \) for each \( A \in \mathcal{A} \), or \( \bigcup \mathcal{A} \) if \( \mu(P) > \mu(A) \) for each \( A \in \mathcal{A} \). Then \( \inf \{ D_\mu(A, P): A \in \mathcal{A} \} = D_\mu(A_0, P). \)

**Proof.** Take a set \( A \in \mathcal{A}. \) We have to show \( D_\mu(A_0, P) \leq D_\mu(A, P). \) Consider two cases:

**Case 1.** \( A_0 \subset A. \) Then

\[
D_\mu(A, P) = \mu(A \cup P) - \mu(P) \geq \mu(A_0 \cup P) - \mu(P) = D_\mu(A_0, P).
\]

**Case 2.** \( A \subset A_0. \) Then by (3) we have

\[
D_\mu(A, P) = \mu(A \cup P) - \mu(A) \geq \mu(A_0 \cup P) - \mu(A_0) = D_\mu(A_0, P).
\]

This completes the proof.

**Fact 8.** Let \( D \) be any metric on \( 2^X \) equivalent to the Hausdorff metric. Then the continuity of a Whitney map \( \mu \) means

\[
\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall A, B \in 2^X: D(A, B) < \delta \Rightarrow |\mu(A) - \mu(B)| < \varepsilon.
\]

If we replace \( D \) by \( D_\mu \) we can put \( \delta = \varepsilon. \)

**Proof.** We have to show \( D_\mu(A, B) < \varepsilon \) implies \( |\mu(A) - \mu(B)| < \varepsilon. \) Assume \( \mu(A) \geq \mu(B). \) Then

\[
\varepsilon > D_\mu(A, B) = \mu(A \cup B) - \mu(B) \geq \mu(A) - \mu(B),
\]

and we are done.

To end the paper we ask some questions connected with condition (3). We say that two Whitney maps \( \mu_1 \) and \( \mu_2 \) are equivalent if for every \( t \) there exist \( t' \) and \( t'' \) such that \( \mu_1^{-1}(t) \) is homeomorphic to \( \mu_2^{-1}(t') \) and \( \mu_2^{-1}(t) \) is homeomorphic to \( \mu_1^{-1}(t''). \)
Question 9. Given any Whitney map $\mu_1$ is there a Whitney map $\mu_2$ which is equivalent to $\mu_1$ and satisfies (3)?

Question 10. Given any continuum $X$ and any Whitney map $\mu: 2^X \to [0, \mu(X)]$ does there exist a homeomorphism $h$ from $[0, \mu(X)]$ into $[0, \infty)$ such that $h \circ \mu$ is a Whitney map satisfying (3)?

References