Abstract. We consider the condition when bounded cohomology injects into ordinary cohomology and prove the vanishing of bounded cohomology of the group of all compactly supported homeomorphisms of $\mathbb{R}^n$.

Introduction. In this note we consider relations among bounded cohomology, ordinary real cohomology and $l^1$ homology of spaces or groups. In particular we present a necessary and sufficient condition under which bounded cohomology injects into ordinary cohomology and by using it prove the vanishing of bounded cohomology and $l^1$ homology of $\text{Homeo}_c \mathbb{R}^n$, the group of all homeomorphisms of $\mathbb{R}^n$ with compact support. We also determine the second bounded cohomology of $\text{SL}_2 \mathbb{R}$.

1. Bounded cohomology. Let us quickly review the theory of bounded cohomology developed by Gromov [2] (see also Brooks [1] and Mitsumatsu [5]). Let $X$ be a topological space and let $\mathcal{C}_*(X) = \{C_q(X), \partial_q\}$ be the singular chain complex of $X$ with real coefficients. Define a norm on $C_q(X)$ by $||\Sigma_{i=1}^n a_i\sigma_i|| = \Sigma_{i=1}^n |a_i|$. The differentials $\partial_q$ are then bounded linear operators.

Let $\mathcal{C}_b^q(X) = \{C_b^q(X), \partial_q\}$ be the norm completion of $\mathcal{C}_q(X)$. Thus $C_b^q(X) = \{\Sigma_{i=1}^n a_i\sigma_i | \Sigma_{i=1}^n |a_i| < \infty\}$ is a Banach space. Passing to the dual Banach spaces, we obtain a cochain complex $\mathcal{C}_b^*(X) = \{C_b^q(X), \delta_q\}$. It is a subcomplex of the ordinary singular cochain complex consisting of bounded cochains. The homology of $\mathcal{C}_b^*(X)$, denoted by $H_b^*(X)$, is called $l^1$ homology of $X$ and the cohomology of $\mathcal{C}_b^*(X)$, denoted by $H_b^*(X)$, is called bounded cohomology of $X$. The inclusions induce homomorphisms $H_*(X) \to H_b^*(X)$ and $H_b^*(X) \to H^*(X)$.

Since the image of a bounded operator is not necessarily a closed subspace, it may happen that the pseudonorms induced on $H_b^*(X)$ or $H_b^*(X)$ are not norms. Following Mitsumatsu [5], we define $\widetilde{H}_b^*(X)$ (resp. $\widetilde{H}_b^*(X)$) to be the quotient of $H_b^*(X)$ (resp. $H_b^*(X)$) by the subspace of pseudonorm zero. In other words, $\widetilde{H}_b^*(X) = Z_b^*(X) / \overline{B}_b^*(X)$ and $\widetilde{H}_b^*(X) = Z_b^*(X) / \overline{B}_b^*(X)$, where $Z$ or $B$ denotes the spaces of (co)cycles or (co)boundaries of the corresponding complex and $\overline{B}$ denotes the closure of $B$. Notice that $\widetilde{H}_b^*(X)$ and $\widetilde{H}_b^*(X)$ are Banach spaces. There is a...
surjective homomorphism $\overline{H}_q^l(X) \to (\overline{H}_q^l(X))'$, where $'$ denotes the dual Banach space.

Now for a group $G$, starting from the real chain complex $\mathcal{C}_*(G) = \{C_q(G), \delta_q\}$, similar constructions as above are made, yielding $l^1$ homology $H_q^l(G)$ and bounded cohomology $H_q^*(G)$ of $G$.

2. Uniform boundary condition. A chain complex is called normed if each chain group is a normed linear space over $\mathbb{R}$ and each differential is a bounded linear operator.

**Definition 2.1.** A normed chain complex $\mathcal{C}_*$ is said to satisfy $q$ uniform boundary condition (q-UBC, for short) if there exists a number $K > 0$ such that for any boundary $z \in B_q$, there is a chain $c \in C_{q+1}$ satisfying $\partial c = z$ and $||c|| \leq K ||z||$.

**Definition 2.2.** (i) A topological space or a group is said to satisfy $q$-UBC if its ordinary chain complex satisfies q-UBC.

(ii) It is said to satisfy $q$-UBC if its $l^1$ chain complex satisfies $q$-UBC.

**Theorem 2.3.** For spaces or groups, the following conditions are equivalent:

(i) $q$-UBC$^l$.

(ii) $B_q^l$ is closed in $C_q^l$.

(iii) $\overline{H}_q^l = H_q^l$.

(iv) $\overline{H}_q^{l+1} = H_q^{l+1}$.

(v) The surjective homomorphism $H_{q+1}^l \to (\overline{H}_q^{l+1})'$ is injective.

**Proof.** (i) is clearly equivalent to that the bijection $C_{q+1}^l/Z_{q+1}^l \to B_q^l$ has a bounded inverse, that is, they are homeomorphic. Since $C_{q+1}^l/Z_{q+1}^l$ is a Banach space, it follows that $B_q^l$ is also a Banach space and hence (ii) holds. Conversely if (ii) is satisfied, then $B_q^l$ is a Banach space and the open mapping theorem (cf. [11]) applied to the above bijection implies (i).

(ii) $\Rightarrow$ (iii) is clear.

(iii) $\Rightarrow$ (iv) is a direct consequence of the closed range theorem.

(i) $\Rightarrow$ (v). Take $f \in Z_{q+1}^l$ so that $f(z) = 0$ for all $z \in Z_{q+1}^l$. By (i), the map $B_q^l \hookrightarrow C_{q+1}^l/Z_{q+1}^l \to \mathbb{R}$ is bounded and thus by the Hahn-Banach theorem has an extension to $C_q^l$. This shows (v).

(v) $\Rightarrow$ (iv). It suffices to show $\text{Ker}(H_{q+1}^l \to \overline{H}_{q+1}^l) \subset \text{Ker}(H_{q+1}^l \to (\overline{H}_{q+1}^l)'$). Let $f \in B_q^{q+1}$. Then $f = \lim_{i \to \infty} f_i$ ($f_i \in B_q^{q+1}$). Thus for any $z \in Z_{q+1}^l$, we have $f(z) = \lim_{i \to \infty} f_i(z) = 0$. This completes the proof. Q.E.D.

**Corollary 2.4.** (i) If $\overline{H}_q^l = H_q^l$ and $\overline{H}_q^{l+1} = 0$, then $H_q^{q+1} = 0$.

(ii) If $\overline{H}_q^{q+1} = H_q^{q+1}$ and $\overline{H}_q^l = 0$, then $H_q^l = 0$.

(iii) The reduced bounded cohomology $\tilde{H}_q^*$ vanishes if and only if the reduced $l^1$ homology $\tilde{H}_q^l$ also vanishes.

In Brooks [1] and Gromov [2], it is shown that the reduced bounded cohomology vanishes for spaces with amenable $\pi_1$. This, combined with the above corollary, gives

**Corollary 2.5.** If $\pi_1(X)$ is amenable, then $\tilde{H}_q^l(X) = 0$. 
Remark 2.6. It seems plausible that the $l^1$ homology of a space depends only on its fundamental group. But we do not have a proof.

Corollary 2.7. For any space or a group, we have

(i) $H^1_l = H^1_b = 0$.
(ii) $H^2_b = H^2_b$. Equivalently, $H^2_b$ is a Banach space.

Proof. First we deal with a group $G$. In [5], Mitsumatsu constructed for $g \in G$,

$$S(g) = \sum_{k=1}^{\infty} \frac{1}{2^k} (g^k, g^k) \in C^1_l(G).$$

Clearly, $\|S(g)\| = 1$ and $\partial S(g) = g$, showing that $H^1_l(G) = 0$ and that $G$ satisfies 1-UBC'. From this follows the corollary. Notice that 0-UBC' is always satisfied.

For spaces, according to Gromov [2], $H^2_b(X)$ is isometric to $H^2_b(\pi_1(X))$. This shows (ii). Also it is known that $H^1_l(X) = 0$ [2]. Thus (i) follows from Corollary 2.4. Q.E.D.

We have shown that 1-UBC' is always true. It would be interesting to determine whether 0-UBC' always holds or not.

Next, we investigate $q$-UBC for spaces or groups.

Theorem 2.8. The following conditions are equivalent:

(i) $q$-UBC.
(ii) $q$-UBC' and $Z_{q+1}$ is dense in $Z^1_{q+1}$.
(iii) The homomorphism $H^1_{q+1} \to H^{q+1}$ is injective.

Proof. (i) $\Rightarrow$ (ii). We first prove $q$-UBC'. Clearly $B_q$ is dense in $B^1_q$. Further a standard argument shows that for any $z \in B^1_q$, there exist $z_i \in B_q$ such that $\sum_{i=1}^{\infty} z_i = z$ and $\sum_{i=1}^{\infty} \|x_i\| \leq (1 + \varepsilon) \|z\|$. Now by $q$-UBC, one can choose $c_i \in C_{q+1}$ such that $\partial c_i = z_i$ and $\|c_i\| \leq K \|z_i\|$. Let $c = \sum_{i=1}^{\infty} c_i \in C^1_{q+1}$. Then $\partial c = z$ and $\|c\| \leq (1 + \varepsilon) K \|z\|$.

Next we prove that $Z_{q+1}$ is dense in $Z^1_{q+1}$. Take $z \in Z^1_{q+1}$ and let $z = \lim_{i \to \infty} c_i$ ($c_i \in C_{q+1}$). Choose an element $d_i \in C_{q+1}$ such that $\partial d_i = -\partial c_i$ and $\|d_i\| \leq K \|\partial c_i\|$. Then we have

$$\|\partial c_i\| = \|\partial (c_i - z)\| \leq \|\partial\| \|c_i - z\| = (q + 2) \|c_i - z\|.$$

Hence $\|d_i\| \leq (q + 2) K \|c_i - z\|$. Now $c_i + d_i \in Z_{q+1}$ and $c_i + d_i \to z$.

(ii) $\Rightarrow$ (i). This is left to the reader.

(ii) $\Rightarrow$ (iii). Take $f \in Z^1_{q+1}$ such that $[f] = 0$ in $H^{q+1}$, that is, $f(z) = 0$ for any $z \in Z_{q+1}$. Then $f(z) = 0$ for any $z \in Z^1_{q+1}$. Thus $f$ induces a bounded map $\hat{f}: C^1_{q+1}/Z^1_{q+1} \to \mathbb{R}$. Now $q$-UBC' implies that the bijection $C^1_{q+1}/Z^1_{q+1} \to B^1_q$ has a bounded inverse. Compose it with $\hat{f}$ and extend to the whole of $C^1_q$ by the Hahn-Banach theorem. This shows (iii).

(iii) $\Rightarrow$ (ii). Notice that the map $H^1_{q+1} \to H^{q+1}$ is a composite of maps $H^1_{q+1} \to \overline{H^1_{q+1}} \to H^{q+1}$. Injectivity of the first map implies $q$-UBC' by Theorem 2.3. Next, clearly (iii) implies that the image of the map $H^1_{q+1} \to \overline{H^1_{q+1}}$ is dense. From this the denseness of $Z_{q+1}$ follows easily. Q.E.D.
Definition 2.9. A group \( G \) is said to be uniformly perfect if for some \( N > 0 \), any \( g \in G \) can be represented as a product of at most \( N \) commutators.

Lemma 2.10. If a group \( G \) is uniformly perfect, then it satisfies 1-UBC.

Proof. Notice that for \( f, g, h \in G \),
\[
\partial \left( (f_1, f_2) + (f_1 f_2, f_3) + \cdots + (f_1 f_2 \cdots f_{N-1}, f_N) \right) = (f_1) + (f_2) + \cdots + (f_N) - (f_1 f_2 \cdots f_N)
\]
and
\[
\partial \left( [[g, h], h] + (ghg^{-1}, g) - (g, h) \right) = ([g, h])
\]
where ( ) denotes a chain and [ ] a commutator. This shows for all \( f \in G \) there exists \( c \in C_2(G) \) such that \( \partial c = (f) \) and \( ||c|| \leq 4N - 1 \). That is, \( G \) satisfies 1-UBC. Q.E.D.

Corollary 2.11. If \( G \) is uniformly perfect, then the map \( H^2_b(G) \to H^2(G) \) is injective.

As applications, we shall compute \( H^2_b \) for some groups.

Example 2.12. The group of all orientation preserving homeomorphisms of \( S^1 \), denoted by \( \text{Homeo}^+(S^1) \), is uniformly perfect. In fact any element is a product of two homeomorphisms with compact support, which are commutators by Mather [4]. Thus \( H^2_b \) injects into \( H^2 \). Now it is a consequence of Thurston’s general result [8] that \( H^*(\text{Homeo}^+(S^1); \mathbb{Z}) \cong \mathbb{Z}[\chi] \), where \( \chi \in H^2 \) is the Euler class. Now \( H^*_b \) is mapped onto real cohomology, because \( \chi \) can be represented by a bounded cocycle (see Morita [6]). Hence we have \( H^2_b(\text{Homeo}^+(S^1)) \cong \mathbb{R} \).

Example 2.13. Sah and Wagoner [7] have calculated second homology of certain Lie groups (considered to be discrete groups). Combined with our result, this gives information about \( H^2_b \). For example, \( \text{SL}_2 \mathbb{R} \) is uniformly perfect (see Wood [10]) and \( H_2(\text{SL}_2 \mathbb{R}; \mathbb{Z}) \) is isomorphic to \( \mathbb{Z} \oplus A \), where \( A \) is a certain \( \mathbb{Q} \)-vector space. \( \mathbb{Z} \) is detected by the “volume class” \( \in H^2(\text{SL}_2 \mathbb{R}) \), which is a bounded cohomology class. Any element of \( A \) is supported on a torus (see [7] and Tsuboi [9]). From these, we can conclude \( H^2_b(\text{SL}_2 \mathbb{R}) \cong H^2_b(\text{PSL}_2 \mathbb{R}) \cong \mathbb{R} \).

3. Bounded cohomology of \( \text{Homeo}_K(\mathbb{R}^n) \). In this section we prove the vanishing of bounded cohomology and \( l^1 \) homology of \( \text{Homeo}_K(\mathbb{R}^n) \), the group of all homeomorphisms of \( \mathbb{R}^n \) with compact support.

Theorem 3.1. For \( q > 0 \),
\[
H^q_b(\text{Homeo}_K(\mathbb{R}^n)) = H^q_{l^1}(\text{Homeo}_K(\mathbb{R}^n)) = 0.
\]

Our argument is a refinement of Mather’s proof of the acyclicity of \( \text{Homeo}_K(\mathbb{R}^n) \). In the sequel we follow Mather [4]. We write \( G = \text{Homeo}_K(\mathbb{R}^n) \) and \( G^i = \{ g \in G : \text{supp } g \subset \text{Int } D^n_i \} \) (\( i = 1, 2, 3 \)), where \( D^n \) is the unit ball. Inclusions are denoted by \( i^1 : G^1 \to G^2 \), \( i^2 : G^2 \to G^3 \), \( i = i^2i^1 \) and \( i : G^1 \to G \). Let \( C_q^i \) and \( C_q^0 \) be the chain complex of \( G \) and \( G^i \). \( Z_q^i \) and \( Z_q^0 \) (resp. \( B_q^i \) and \( B_q^0 \)) denote the cycle group (resp.
boundary group) of the corresponding complexes. In fact, $Z_q = B_q$ and $Z'_q = B'_q$ by the acyclicity of the group.

We shall prove inductively the existence of bounded linear operators $S_q : B^1_q \rightarrow C_d^{q+1}$ such that $\partial_{q+1}S_q = \iota_*$. Let us show first that this suffices for our purpose. We have only to show $q$-UBC for $G$, because the acyclicity of $G$, together with Theorem 2.8 and Corollary 2.4 yields Theorem 3.1. Take $z \in B^1_q$. Choose $\varphi \in G$ such that $\varphi$ is the identity on $\text{supp} \, z$ and maps $\text{supp} \, S_q(z)$ into $\text{Int} \, D^n$. Define $I_{\varphi} : G \rightarrow G$ by $I_{\varphi}(g) = \varphi g \varphi^{-1}$. Then we have $I_{\varphi}^*(S_q(z)) \in C_d^{q+1}$, $||I_{\varphi}^*(S_q(z))|| = ||S_q(z)|| \leq ||z||$ and $\partial(I_{\varphi}^*(S_q(z))) = I_{\varphi}^*(\partial S_q(z)) = I_{\varphi}^*(z) = z$. This proves $q$-UBC for $G^1$, hence for $G$.

$S_1$ is constructed in an elementary fashion as follows. Choose $k \in G$ such that $k(3D^n) \cap 3D^n = \emptyset$ and that $k'(3D^n)$ tends to one point as $i \rightarrow \infty$. Define $\psi_1 : G^3 \rightarrow G$ by $\psi_1(g) = \sum_{i=1}^n k^ig^{-i}$ and let $\psi_0(g) = k^{-1}\psi_1(g)k$. Then for any $g \in G^3$, $\text{supp} \, g \cap \text{supp} \, \psi_1(g) = \emptyset$ and $\psi_0(g) = g\psi_1(g)$. The restrictions of $\psi_1$ to $G^1$ are denoted by the same letter. Now we define a bounded linear map $S_1 : B^1_1 \rightarrow C_2$ by

$$S_1(g) = (k, g) - (\psi_1(g), k) + (kg, \psi_1(g)).$$

Direct computation shows $\partial_2S_1 = \iota_*$.

Next we assume there exist $S^j_1 : B^j_1 \rightarrow C_d^{j+1}$ and $S^j_2 : B^j_2 \rightarrow C_d^{j+1}$ for $0 \leq j \leq q - 1$ and construct $S_q : B^1_q \rightarrow C_d^{q+1}$. Let $\alpha : C_*(G \times G) \rightarrow C_* \otimes C_*$ (resp. $\beta : C_* \otimes C_* \rightarrow C_*(G \times G)$ be the Alexander-Whitney map (resp. Eilenberg-Mac Lane map) (see Mac Lane [3]). Those maps for $G^i$ are also denoted by the same letters. They are functorial and if we give a norm to $C_* \otimes C_*$ in a canonical manner, they are bounded linear.

For each $z \in B^1_q$, define $D(z) = \alpha_{\Delta \ast}z - (z \otimes 1 + 1 \otimes z)$, where $\Delta : G^1 \rightarrow G^1 \times G^1$ is the diagonal map. Then $D(z) \in Z_q'(C^1 \otimes C^1) = Z_q(C^1 \otimes C^1) \cap \Sigma_{i=1}^n C_i \otimes C_i$. $D$ is bounded linear. Now let $Z^1$ and $\overline{B}^1$ be the chain complexes (with trivial differential) defined by $(Z^1)_q = Z^1_q$ and $(\overline{B}^1)_q = B^1_q$. Then we have the following commutative diagram, whose horizontal sequences are all exact.

$$
\begin{array}{cccccc}
0 & \rightarrow & (C^1 \otimes Z^1)_{q+1} & \rightarrow & (C^1 \otimes C^1)_{q+1} & \rightarrow & (C^1 \otimes \overline{B}^1)_{q+1} & \rightarrow & 0 \\
& \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
0 & \rightarrow & (C^1 \otimes Z^1)_q & \rightarrow & (C^1 \otimes C^1)_q & \rightarrow & (C^1 \otimes \overline{B}^1)_q & \rightarrow & 0 \\
& \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
0 & \rightarrow & (C^1 \otimes Z^1)_{q-1} & \rightarrow & (C^1 \otimes F^1)_{q-1} & \rightarrow & (C^1 \otimes \overline{B}^1)_{q-1} & \rightarrow & 0 \\
\end{array}
$$

Analogous diagrams are considered for $G^2$ and $G^3$. They are combined by $i_*^1$ and $i_*^2$. Notice that $Z_q'(C \otimes \overline{B}^1) = (Z^1 \otimes \overline{B}^1)_q$. Hence by the induction assumption we can consider

$$(S^1 \otimes S^1)(1 \otimes \partial)D(z) = (S^1 \otimes S^1)D(z) \in (C^2 \otimes C^2)_{q+1}.$$

Let $u = (i_*^1 \otimes i_*^2)D(z) - \partial(S^1 \otimes S^1)D(z) \in (C^2 \otimes C^2)_q$. Direct computation shows that $(1 \otimes \partial)u = 0$. Thus we have $u \in (C^2 \otimes Z^2)_q$. Also we have $\partial u = 0$. That is,

$$u \in Z_q'(C^2 \otimes Z^2) = (Z^2 \otimes Z^2)_q.$$
Therefore by the induction assumption, we can consider
\[(S^2 \otimes (\iota_*^2 - S^2\partial))u \in (C^3 \otimes C^3)_{q+1}^+\]
Again direct computation shows that
\[\partial(S^2 \otimes (\iota_*^2 - S^2\partial))u = (\iota_*^2 \otimes \iota_*^2)u.\]
Hence we have \((\iota_* \otimes \iota_*)D(z) = \partial(ED(z)),\)
where
\[E = (\iota_*^2 S^1 \otimes \iota_*^2 S^2\partial) + (S^2 \otimes (\iota_*^2 - S^2\partial))(\iota_*^2 - \partial(S^1 \otimes S^1\partial)).\]
For any nonnegative integer \(q\), \(E : Z_q'(G^1 \otimes C^1) \to (C^3 \otimes C^3)_{q+1}^+\) is a bounded linear map. We can now write
\[(\iota_* \otimes \iota_*)\alpha \Delta_* z = (\iota_* \otimes \iota_*)z \otimes 1 + \partial(ED(z)) + (\iota_* \otimes \iota_*)(1 \otimes z).\]
Let \(\eta : G^3 \otimes G^3 \to G\) be the homomorphism given by \(\eta(g, h) = g\psi_1(h)\). Applying \(\eta_*\beta\), we get
\[\eta_*(i \times i)_* \beta \Delta_* z = \iota_* z + \eta_* \beta \partial(ED(z)) + \chi_{1*} z.\]
As is well known, \(\beta \alpha\) is chain homotopic to the identity. Namely there is a linear map \(\Phi : C_*(G^1 \times G^1) \to C_{*+1}^+((G^1 \times G^1))\) such that \(\beta \alpha - \text{id} = \partial \Phi + \Phi \partial\). It is easy to show that we can choose \(\Phi\) as a bounded linear map. Now \(\beta \alpha \Delta_* z = \Delta_* z + \partial \Phi(\Delta_* z)\). Because \(\eta \Delta = \psi_0\), we have
\[\psi_0^* z + \eta_*(i \times i)_* \partial \Phi \Delta_* z = \iota_* z + \eta_* \beta \partial ED(z) = \psi_1^* z.\]
Two homomorphisms \(\psi_0\) and \(\psi_1\) are conjugate and thus \(\psi_0^*\) is chain homotopic to \(\psi_1^*\). Namely there is a linear map \(\Theta : C_{*+1} \to C_{*+1}^+\) such that \(\psi_1^* - \psi_0^* = \partial \Theta + \Theta \partial\). Here we can also choose \(\Theta\) to be bounded linear. Finally we obtain \(\iota_* = \partial_{q+1}^+ S_q\), where
\[S_q = \eta_*(i \times i)_* \Phi \Delta_* - \Theta - \eta_* \beta ED.\]
This completes the proof.

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