THE CATENARIAN PROPERTY OF POWER SERIES RINGS
OVER A PRÜFER DOMAIN

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ABSTRACT. Let $D$ be a Prüfer domain that has the SFT-property. It is shown that the power series ring $D[[x]]$ is catenarian. If $n > 1$ and $\dim D > 1$ then the ring $D[[x_1, \ldots, x_n]]$ is not catenarian.

1. Introduction. Recall that a ring $R$ is a catenary ring if for each saturated chain $P_0 < P_1 < \cdots < P_n$ of prime ideals of $R$ we have $\text{rank}(P_n/P_0) = n$. In particular, an integral domain $D$ is catenarian if, for every prime ideal $P$ of $D$, the length of every saturated chain of prime ideals between $(0)$ and $P$ equals $\text{rank} P$. Based on the work of Ratliff in [9] and Seydi in [10], Lequain has shown in [7] that if $D$ is a Noetherian integral domain then the domains $D[x]$, $D[x_1, \ldots, x_n]$, $D[[x]]$, and $D[[x_1, \ldots, x_n]]$ are simultaneously catenarian.

In [4] and [5] it is shown that if $D$ is a finite-dimensional Bezout domain then $D[x]$ is catenarian. More recently, De Souza Doering and Lequain have shown in [8] that if $D$ is a finite dimensional Prüfer domain then $D[x]$ is catenarian. Finally, Bouvier and Fontana have shown in [6] that if $D$ is a locally finite-dimensional Prüfer domain then $D[x_1, \ldots, x_n]$ is catenarian for each $n > 1$. In view of the above referenced results for Noetherian integral domains, it is natural to inquire about the catenarian properties of $D[[x]]$ and $D[[x_1, \ldots, x_n]]$ when $D$ is a Prüfer domain.

In [1] the author has defined an ideal $A$ of a ring $R$ to be of strong finite type (or an SFT-ideal) provided there is a finitely generated ideal $B \subset A$ and a positive integer $k$ such that $a^k \in B$ for each $a \in A$. The ring $R$ satisfies the SFT-property provided each ideal of $R$ is an SFT-ideal. It is shown in [1] (cf. the proof of [1, Theorem 1]) that if a ring $R$ does not have the SFT-property then there exists an infinite chain of prime ideals in $R[[X]]$. We must, therefore, restrict our attention to Prüfer domains with the SFT-property. Such domains have already been studied in [2 and 3].

2. The catenarian property in $D[[x]]$. Throughout this section, $D$ denotes a Prüfer domain with the SFT-property and $K$ denotes the quotient field of $D$. We let $X_n = \{x_1, \ldots, x_n\}$ and we write $D[[X_n]]$ to denote $D[[x_1, \ldots, x_n]]$.

**Lemma 1.** Let $Q_1$ and $Q_2$ be prime ideals of $D[[X_n]]$ such that $Q_2 < Q_1$, $\text{rank}(Q_1/Q_2) = 1$, and $Q_i \cap D = P_i$, $i = 1, 2$. Then either $P_1 = P_2$ or $\text{rank}(P_1/P_2) = 1$ and $Q_1 \subset P_1 + (X_n)$.

**Proof.** Suppose that $P_2 < P_1$, let $p \in P_1 \setminus P_2$ (cf. [2, Lemma 2.7]), and let $W$ be a valuation overring of $D[[X_n]]$ with prime ideals $N_2 < N_1$ such that $N_i \cap D[[X_n]] = Q_i$, $i = 1, 2$, $N_1$ is maximal in $W$, and $N_1 = \text{rad}(pW)$. Since $V = W \cap K$ is a
valuation overring of \( D \) centered on \( P_1 \) we must have \( V = DP_1 \). Assume there exists a prime ideal \( P \) of \( D \) such that \( P_2 < P < P_1 \). Then \( PV \subset \bigcap_{k=1}^{\infty} p^k V \) so \( PW \subset \bigcap_{k=1}^{\infty} p^k W \). If we set \( N = \bigcap_{k=1}^{\infty} p^k W \) then \( N \) is a prime ideal of \( W \) and, since \( P \subset N \cap D < P_1 \), it follows that \( Q_2 < N \cap D[[X_n]] < Q_1 \). But this contradicts our assumption that \( \text{rank}(Q_1/Q_2) = 1 \), so we conclude that \( \text{rank}(P_1/P_2) = 1 \).

Let \( S \) denote the set of invertible ideals \( A \) of \( D \) such that \( P_1 < A \). As in [3, p. 901], we see that \( S \) is a multiplicatively closed set of invertible ideals and \( D_S = \{ \xi \in K | \xi A \subset D \text{ for some } A \in S \} \) is a flat overring of \( D \) with prime ideals \( \{ PD_S \mid P \text{ is a prime ideal of } D \text{ and } P \neq P_1 \} \). In particular, \( P_1 D_S \) is a maximal ideal of \( D_S \) and, since \( P_1 D_S \cap D = P_1 \) and \( P_1 D_S \subset D \), we have \( P_1 = P_1 D_S \). Following the proof of [3, Theorem 3.6] we see that \( W \supset D_S[[X_n]] \).

Set \( P_1 = N_1 \cap D_S[[X_n]] \). Then \( P_1 \cap D_S = P_1 D_S \), so \( P_1 \subset [P_1 + (X_n)]D_S[[X_n]] \). Therefore, \( Q_1 = P_1 \cap D[[X_n]] \subset P_1 + (X_n) \).

For a prime ideal \( Q \) of \( D[[x]] \), let \( \mathcal{A}(Q) \) denote the set of prime ideals \( Q' \) of \( D[[x]] \) such that \( Q < Q' \) and \( \text{rank}(Q'/Q) = 1 \). Similarly, \( \mathcal{B}(Q) \) denotes the set of prime ideals \( Q' \) of \( D[[x]] \) such that \( Q' < Q \) and \( \text{rank}(Q/Q') = 1 \). Finally, \( I(Q) \) denotes the ideal of \( D \) consisting of the constant terms of elements of \( Q \).

**Lemma 2.** Let \( Q \) be a prime ideal of \( D[[x]] \) such that \( Q \cap D = P \), \( P[[x]] < Q \), and \( Q \neq P + (x) \). Then

1. \( \mathcal{A}(Q) \) is a finite set and \( \mathcal{A}(Q) = \{ P' + (x) \mid P' \text{ is minimal over } I(Q) \} \subset \{ P' + (x) \mid \text{rank}(P'/P) = 1 \} \) and
2. \( P[[x]] \) is the unique element in \( \mathcal{B}(Q) \).

**Proof.** Let \( Q' \) be a prime ideal of \( D[[x]] \) and set \( P' = Q' \cap D \). Then \( P'[[x]] \subset Q' \) by [1, Theorem 1] and, by applying Lemma 3.5 of [3] to \( (D/P')[[x]] \), we see that either \( Q' = P'[[x]] \) or \( \text{rank}(Q'/P'[[x]]) = 1 \). If \( Q' \in \mathcal{A}(Q) \) it follows that \( Q' < P' \), so, by Lemma 1, \( \text{rank}(P'/P) = 1 \) and \( Q' \subset P' + (x) \). But \( Q' \neq P'[[x]] \) by [2, Corollary 3.6] so \( Q' = P' + (x) \). Since \( Q < Q' \) it is necessary that \( I(Q) \subset P' \). But \( I(Q) \notin P' \) (otherwise we would have \( Q \subset P + (x) \)), so \( P' \) is minimal over \( I(Q) \).

Conversely, let \( P' \) be a prime ideal of \( D \) such that \( P' \) is minimal over \( I(Q) \). Then \( Q < P' + (x) \), so there exists a prime ideal \( Q'' \) of \( D[[x]] \) such that \( Q < Q'' \subset P' + (x) \) and \( Q'' \in \mathcal{A}(Q) \). As we have just shown, \( Q'' = P'' + (x) \) for some prime ideal \( P'' \) of \( D \) such that \( \text{rank}(P''/P) = 1 \) and \( P'' \) is minimal over \( I(Q) \). It follows that \( P' = P'' \) and, hence, that \( P' + (x) \in \mathcal{A}(Q) \). The finiteness of \( \mathcal{A}(Q) \) follows from Corollary 2.6 of [2].

Now suppose that \( Q' \in \mathcal{B}(Q) \) and \( P' = Q' \cap D \). Since \( Q \notin P + (x) \), it follows from Lemma 1 that \( P = P' \). Consequently, \( Q' = P[[x]] \).

**Lemma 3.** Let \( P \) be a nonzero prime ideal of \( D \) and let \( (0) < Q_1 < \cdots < Q_s < P + (x) \) be a saturated chain of prime ideals in \( D[[x]] \). Then \( s = \text{rank } P \).

**Proof.** Suppose that \( s > 1 \) and let \( P_1 \) be the unique rank one prime ideal of \( D \) that is contained in \( P \). Either \( Q_1 = P_1[[x]] \) or \( Q_1 \cap D = (0) \). Since \( Q_2 \cap D \neq (0) \) by [3, Lemma 3.5], \( Q_1 \cap D = (0) \) implies, by Lemma 1, that \( Q_2 \cap D = P_1 \). Thus, since \( \text{rank } P_1[[x]] = 1 \), in either case we have that \( Q_2 > P_1[[x]] \). Since \( P + (x) > Q_2 \) by assumption, it follows that \( P \neq P_1 \); that is, \( \text{rank } P \geq 2 \). Thus, if \( \text{rank } P = 1 \), then \( s = 1 \). On the other hand, if \( \text{rank } P > 1 \) we may pass to \( D/P_1 \) and argue, by induction, that \( s = \text{rank } P \).
**THEOREM 4.** If $D$ is a Prüfer domain with the SFT-property, then $D[[x]]$ is catenarian.

**Proof.** Let $Q$ be a nonzero prime ideal of $D[[x]]$ with $P = Q \cap D$. If $Q = P[[x]]$ then, by [2, Corollary 3.6], there is a unique chain of prime ideals between $(0)$ and $Q$. Likewise, if $Q > P[[x]]$ but $Q \neq P + (x)$, then, by Lemma 2(2) and [2, Corollary 3.6], there is a unique chain of prime ideals between $(0)$ and $Q$. The remaining case, $Q = P + (x)$, has already been handled in Lemma 3.

If $\dim D = 1$ then $D$ is a Dedekind domain (cf. [2, p. 4]) and hence, by [7, Theorem 2.6], $D[[X_n]]$ is catenarian for each $n$. If $\dim D = m$ then, by [3, Theorem 3.6], $\dim D[[X_n]] = mn + 1$. Indeed, if $M$ is a maximal ideal of $D$ and rank $M = m$, then rank($M + (X_n)$) = $mn + 1$. But rank($x_n$) = 1 and

$$\text{rank}((M + (X_n))/(x_n)) = \text{rank}(M + (X_{n-1})) = m(n-1) + 1,$$

so if $m > 1$ then each saturated chain of the form $(0) < (x_n) < \cdots < M + (X_n)$ has length at most $m(n-1) + 2 < mn + 1$. Thus, $D[[X_n]]$ is not catenarian.

**3. The valuation ring case.** A valuation ring $V$ has the SFT-property if and only if it is a discrete valuation ring. (A valuation ring $V$ is discrete provided $P \neq P^2$ for each prime ideal $P$ of $V$. Cf. Lemma 2.7 and Proposition 3.1 of [2].) The results of §2 allow us to give a fairly complete description of the prime spectrum of $V[[x]]$ when $V$ is a discrete valuation ring. First, we restate and strengthen Lemma 2 as follows.

**Proposition 5.** Let $V$ be a discrete valuation ring and let $Q$ be a prime ideal of $V[[x]]$. If $Q \cap V = P$, $P[[x]] < Q$, and $Q \neq P + (x)$, then

1. $A(Q) = \{P' + (x)\}$ where $P'$ is the unique prime ideal of $V$ such that $P < P'$ and rank($P'/P$) = 1,
2. $B(Q) = \{P[[x]]\}$,
3. $Q$ is principal.

**Proof.** (1) and (2) are immediate from Lemma 2. We first prove (3) for the case $P = (0)$. In this case rank $P' = 1$, and if we set $W = V_{P'}$, then $W[[x]]_{V \setminus \{0\}} = V[[x]]_{V \setminus \{0\}}$. It follows that $Q' = QV[[x]]_{V \setminus \{0\}}$ is a prime ideal of $W[[x]]$ such that $Q' \cap W = (0)$ and $Q' \cap V[[x]] = Q$. But $W[[x]]$ is a unique factorization domain so $Q'$ is principal.

Suppose that $Q' = f'W[[x]]$, where $f' = \sum_{i=0}^{\infty} w_i x^i$. Since $Q' \not\subset (PW)[[x]]$, some $w_i, i \geq 1$, is a unit in $W$. Thus, we may write $f' = f'_1 + x^k u$ where $k \geq 1$, $f'_1 \in (PW)[[x]]$, and $u$ is a unit in $W[[x]]$. If $f = f' u^{-1} = f'_1 u^{-1} + x^k f_1 = f_1 + x^k$ then $f_1 = f'_1 u^{-1}$ is in $(PW)[[x]] = P[[x]] \subset V[[x]]$. Therefore, $f \in V[[x]]$ and $Q' = fW[[x]]$. Let $q \in Q$ and write $q = fh$ where $h \in W[[x]]$. Then $q = f_1 h + x^k h$ and $f_1 h \in P[[x]] \subset V[[x]]$ so $x^k h = q - f_1 h \in V[[x]]$. It follows that $h \in V[[x]]$ and $Q = fV[[x]]$.

In the general case we may now conclude that $Q = P[[x]] + fV[[x]]$ where, if $f = \sum_{i=0}^{\infty} a_i x^i$, then $a_0 \notin P$. Thus, if $p \in P$ then $p/a_0^n \in V$ for each positive integer $n$. It follows that $Pf^{-1} \subset V[[x]]$ and, hence, that $P[[x]] \subset fV[[x]]$. Therefore, $Q = fV[[x]]$.

**Question.** What is a characterization of the prime elements of $V[[x]]$?
We conclude with a diagram which illustrates the prime spectrum of a three-dimensional discrete valuation ring $V$. Let $(0) = P_0 < P_1 < P_2 < P_3$ be the prime ideals of $V$ and let $f_i$, $0 \leq i \leq 2$, denote arbitrary prime elements of $V[[x]]$ such that $f_i V[[x]] \cap V = P_i$. We use $A \rightarrow B$ to mean $A > B$. We thus have the following diagram of prime ideals in $V[[x]]$.

\[
\begin{array}{cccc}
(x) & \rightarrow & P_1 + (x) & \rightarrow & P_2 + (x) & \rightarrow & P_3 + (x) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\phi V[[x]] & \rightarrow & f_1 V[[x]] & \rightarrow & f_2 V[[x]] & \rightarrow & \phi V[[x]] \\
(0) & \rightarrow & P_1[[x]] & \rightarrow & P_2[[x]] & \rightarrow & P_3[[x]]
\end{array}
\]

REFERENCES


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