THREE-SPACE PROBLEM FOR LOCALLY UNIFORMLY ROTUND RENORMINGS OF BANACH SPACES

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Abstract. If Y is a subspace of a real Banach space X such that X/Y admits an equivalent LUR norm, then X admits an equivalent LUR (strictly convex) norm provided Y also does.

1. Introduction. It may happen that a Banach space with quite complicated structure may possess nice factors through nice subspaces (see e.g. [4, 6, 7]). Thus the question of what properties are shared by the whole space X, if satisfied in both Y ⊂ X and X/Y, is of some interest. Concerning such properties linked with renorming theory, it is known that such a property, for instance, is being isomorphic to a uniformly convex space [4] while it is not the case for the property of being isomorphic to Hilbert space [4, 7], nor for the property to be weakly compactly generated (see e.g. [3]). Recently M. Talagrand proved that this also is not the case for the property of the space admitting an equivalent Gateaux smooth norm [11].

A norm | · | of a Banach space X is called locally uniformly rotund (LUR) if \( \lim |x_n - x| = 0 \) for each \( x_n, x \in X \), for which \( \lim 2|x|^2 + 2|x_n|^2 - |x + x_n|^2 = 0 \).

The result of this paper originated from a more detailed study of the geometry in Day's construction of an LUR norm on \( c_0(\Gamma) \) (see [10]) and, mainly, of its extension to spaces with transfinite Schauder bases in [12]. Some arguments of [5] are used here too.

Partial results connected with our results, namely those for the cases where either Y or X/Y are separable, were proved in [1] and [5].

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2. Result.

Theorem. Let X be a real Banach space and Y be such a subspace of X such that X/Y admits an equivalent LUR norm. Then X admits an equivalent LUR (strictly convex) norm provided that the subspace Y also does.
Proof. (Depends heavily on that in [12].) We shall consider only the case of LUR; the other case can be dealt with similarly (see [5]). Let \( \| \cdot \| \) be an equivalent norm on \( X \), the restriction of which to \( Y \) is LUR. (For a simple construction of such a norm see e.g. [5].) Furthermore, let \( | \cdot | \) denote an equivalent LUR norm on \( X/Y \) which is greater than or equal to the factor norm \( | \cdot |_1 \) on \( X/Y \) given by \( \| \cdot \| \). Denote by \( S_1 = \{ \hat{x} \in X/Y, |\hat{x}| = 1 \} \), where \( \hat{x} \) means the element of \( X/Y \) given by \( x \).

Let \( B : X/Y \to X \) denote the Bartle-Graves continuous selection map (i.e. \( B\hat{x} \in \hat{x} \) —see e.g. [2, p. 86]). For each \( \hat{a} \in S_1 \subset X/Y \), let \( f_\hat{a} \in X^* \) be such that \( f_\hat{a}(B\hat{a}) = 1 \), \( \| f_\hat{a} \| = |\hat{a}|_1^{-1} \), \( f_\hat{a} = 0 \) on \( Y \), and let, for \( x \in X \), \( P_\hat{a}(x) = f_\hat{a}(x) \cdot (B\hat{a}) \). Now, for each \( \hat{a} \in S_1 \subset X/Y \) and each positive integer \( k \), define the following function \( \Phi_{k,\hat{a}} \) on \( X \):

\[
\Phi_{k,\hat{a}}(x) = |\hat{x} + \hat{a}|^2 + k^{-1}(1 + \|P_\hat{a}\|)^{-2}\|x - P_\hat{a}(x)\|^2, \quad \text{for} \ x \in X.
\]

Furthermore, let

\[
\Phi_k(x) = \sup \{ \Phi_{k,\hat{a}}(x), \hat{a} \in S_1 \subset X/Y \}, \quad \text{for} \ x \in X,
\]

and

\[
\Phi(x) = \|x\|^2 + |\hat{x}|^2 + \sum_k 2^{-k}\Phi_k(x), \quad \text{for} \ x \in X.
\]

Finally, let \( \| \cdot \| \) be the Minkowski functional of the set \( \{ x \in X: \Phi(x) + \Phi(-x) \leq 4 \} \).

The functions \( \Phi_{k,\hat{a}}(x) \) will be used to transfer the LUR property of the norms \( \| \cdot \| \) on \( Y \) and \( | \cdot | \) on \( X/Y \) to the whole space \( X \).

It is easy to see that \( \| \cdot \| \) is an equivalent norm on \( X \). We now show that it is LUR. To do this suppose that, for some \( \varepsilon > 0 \), \( x \in X \) and sequence \( \{ x_n \} \) such that

\[
(1) \quad \|x\| = 1 = \|x_n\|, \quad \lim \|x + x_n\| = 2 \quad \text{and} \quad \|x - x_n\| > \varepsilon > 0
\]

and find a contradiction.

Because of the uniform continuity of the function \( \Phi_0(x) = \Phi(x) + \Phi(-x) \) on bounded sets on \( X \), we have from (1), that

\[
\Phi_0(x) = \Phi_0(x_n) = 1, \quad \lim \Phi_0((x + x_n)/2) = 1
\]

and thus

\[
\frac{1}{2}\Phi_0(x) + \frac{1}{2}\Phi_0(x_n) - \Phi_0((x + x_n)/2) \to_n 0,
\]

and, from a convexity argument,

\[
\frac{1}{2}\Phi(x) + \frac{1}{2}\Phi(x_n) - \Phi((x + x_n)/2) \to_n 0. \tag{2}
\]

Again by convexity, (2) implies that

\[
\frac{1}{2}\|x\|^2 + \frac{1}{2}\|x_n\|^2 - \|(x + x_n)/2\|^2 \to_n 0, \tag{3}
\]

\[
\frac{1}{2}|\hat{x}|^2 + \frac{1}{2}|\hat{x}_n|^2 - |(\hat{x} + \hat{x}_n)/2|^2 \to_n 0, \tag{4}
\]

and, for each \( k \),

\[
\frac{1}{2}\Phi_k(x) + \frac{1}{2}\Phi_k(x_n) - \Phi_k((x + x_n)/2) \to_n 0. \tag{5}
\]
and
\( K = \sup \{ \| x_n \| \} < \infty. \)

First, if \( x \in Y \), then \( \hat{x} = 0 \) and, from (5), we have \( \lim |\hat{x}_n| = 0 \). Thus there are \( x'_n \in Y \), with \( \lim \| x_n - x'_n \| = 0 \). Then we have by (3) that
\[ 2\| x \|^2 + 2\| x'_n \|^2 - \| x + x'_n \|^2 \to 0 \]
and, since \( \| \cdot \| \) is LUR on \( Y \), \( \lim \| x'_n - x \| = 0 \), which contradicts (1).

If \( x \notin Y \), let \( t = |\hat{x}|^{-1} > 0 \). Write \( tx = y_0 + s_0, s_0 = B(t\hat{x}), y_0 \in Y \).

From LUR of \( \| \cdot \| \) on \( Y \), there is a \( \delta \in (0, \frac{1}{2}) \) such that whenever \( y \in Y, \| y - y_0 \| \leq \delta, z \in Y, \frac{1}{2}\| y \|^2 + \frac{1}{2}\| z \|^2 - \| (y + z)/2 \|^2 \leq \delta \), then
\( \| y - z \| < \epsilon t/2. \)

Let \( \delta_1 \in (0, \delta) \) be such that whenever \( y \in X, |\hat{y} - t\hat{x}| < \delta_1 \), then
\[ \| P_y \| < \| s_0 \| : |s_0|^{-1} - 1. \]

Put
\[ \delta_2 = \min \{ 10^{-1}(tK + 1)^{-1}\| s_0 \| : |s_0|^{-1} + 2 \} \]

From LUR of \( | \cdot | \) of \( X/Y \), choose \( \delta_3 > 0 \) such that if \( \hat{a} \in S_1 \subset X/Y, |\hat{x} + \hat{a}|^2 \geq (t^{-1} + 1)^2(1 - 4\delta_3) \), then
\( |t\hat{x} - \hat{a}| < \delta_2 \) and \( \| B(t\hat{x}) - B(\hat{a}) \| < \delta_2 \)
(see e.g. [8, p. 343]). Finally choose, in the definition of our norm, an integer \( k \) such that
\( k > \delta_3^{-1}K^{-2} \)
and fix this \( k \) until the end of the proof.

From (5) and (4) we have that
\[ c_n = \frac{1}{2}\Phi_k(x) + \frac{1}{2}\Phi_k(x_n) - \Phi_k((x + x_n)/2) \to 0 \quad \text{and} \quad \lim |\hat{x}_n - \hat{x}| = 0. \]

Let \( \hat{a}_n \in S_1 \subset X/Y \) be such that
\[ d_n = \Phi_k((x + x_n)/2) - \Phi_k,\hat{a}_n((x + x_n)/2) \to 0. \]

Then
\[ c_n \geq \frac{1}{2}\Phi_k,\hat{a}_n(x) + \frac{1}{2}\Phi_k,\hat{a}_n(x_n) - \Phi_k,\hat{a}_n((x + x_n)/2) - d_n = b_n - d_n \]
for some nonnegative \( b_n \), and thus, since \( \lim c_n = \lim d_n = 0 \), we have that \( \lim b_n = 0 \) as well. Therefore
\[ b_n = \frac{1}{2}|\hat{x} + \hat{a}_n|^2 + \frac{1}{2k}(1 + \| P_{\hat{a}_n} \|)^{-2}\| x - P_{\hat{a}_n}(x) \|^2 \]
\[ + \frac{1}{2}|\hat{x}_n + \hat{a}_n|^2 + \frac{1}{2k}(1 + \| P_{\hat{a}_n} \|)^{-2}\| x_n - P_{\hat{a}_n}(x_n) \|^2 \]
\[ - \frac{1}{2}|\hat{x} + \hat{x}_n/2 + \hat{a}_n|^2 - \frac{1}{k}(1 + \| P_{\hat{a}_n} \|)^{-2}\| x + x_n/2 - P_{\hat{a}_n}(x + x_n/2) \|^2 \to 0 \]
and by the convexity argument,

\begin{align}
(1 + \|P_{\alpha_n}\|)^{-1} &\frac{1}{2} \|x - P_{\alpha_n}(x)\|^2 + \frac{1}{2} \|x_n - P_{\alpha_n}(x_n)\|^2 \\
- \frac{1}{2} \left\| \frac{x + x_n}{2} - P_{\alpha_n} \left( \frac{x + x_n}{2} \right) \right\|^2 &\to n \to 0
\end{align}

(12)

We now show that beginning with some index \(n_0\), we have that

\begin{equation}
|x + \hat{\alpha}_n|^2 \geq (t^{-1} + 1)^2(1 - 4\delta_3).
\end{equation}

(13)

For, if it were not the case, then taking \(\hat{\alpha}'_n = t^{-1}\hat{x}\) we would have, for infinitely many \(n\)'s,

\begin{equation}
|x + \hat{\alpha}'_n|^2 > |\hat{x} + \hat{\alpha}_n|^2 + 4\delta_3.
\end{equation}

Then for these \(n\)'s we would have, because of the convexity

\begin{align*}
c_n &\geq \frac{1}{2} |\hat{x} + \hat{\alpha}_n|^2 + \frac{1}{2k} (1 + \|P_{\alpha_n}\|)^{-1} \|\hat{x} - P_{\alpha_n}(x)\|^2 \\
&\quad + \frac{1}{2k} (1 + \|P_{\alpha_n}\|)^{-2} \|x - P_{\alpha_n}(x)\|^2 - \frac{1}{2k} (1 + \|P_{\alpha_n}\|)^{-2} \|x - P_{\alpha_n}x\|^2 \\
&\quad + \frac{1}{2} |\hat{x} + \hat{\alpha}_n|^2 - \frac{1}{2} \left( \frac{x + x_n}{2} - P_{\alpha_n} \left( \frac{x + x_n}{2} \right) \right)^2 - d_n
\end{align*}

(14)

which contradicts \(c_n \to 0, d_n \to 0\). Therefore, beginning with some index \(n_0\), we have that \(|\hat{x} + \hat{\alpha}_n|^2 \geq (t^{-1} + 1)^2(1 - 4\delta_3)\) and hence by (9) and (8) we have that

\begin{equation}
|\hat{\alpha}_n - \hat{x}| < \delta_2 \leq \delta_1, \quad \|B\hat{\alpha}_n - B(t\hat{x})\| < \delta_2 \quad \text{and} \quad \|P_{\alpha_n}\| < \|s_0\| : \|\delta_0\|^{-1} + 1.
\end{equation}

Thus, by (12), for sufficiently large \(n \geq n_0\), we have that

\begin{equation}
\frac{1}{2} \|tx - P_{\alpha_n}(tx)\|^2 + \frac{1}{2} \|tx_n - P_{\alpha_n}(tx_n)\|^2 - \left\| \frac{tx + tx_n}{2} - P_{\alpha_n} \left( \frac{tx + tx_n}{2} \right) \right\|^2 < \delta_2
\end{equation}

(15)

and

\begin{equation}
|\hat{\alpha}_n - \hat{\delta}_0| < \delta_2
\end{equation}

(16)
and
\[ |t\hat{x}_n - t\hat{z}| < \delta_2 \]  
(use (4) together with LUR of $|\cdot|$ on $X/Y$). Fix such an $n$ until the end of our proof.

Now choose an element $z_n \in \hat{a}_n$ such that
\[ \|z_n - tx\| < \delta_2 \]  
(use (16)), and an element $x'_n \in \hat{a}_n$ such that
\[ \|x'_n - tx_n\| < 2\delta_2 \]  
(use (16) and (17)).

Setting $a_n = B\hat{a}_n$, write $x'_n = a_n + u_n, u_n \in Y; z_n = a_n + v_n, v_n \in Y$. Then
\[ x'_n - P_{\hat{a}_n}x'_n = u_n, \quad z_n - P_{\hat{a}_n}z_n = v_n \]
and, since $tx = s_0 + y_0$ and $z_n = a_n + v_n$, we have that
\[ \|v_n - y_0\| \leq \|tx - z_n\| + \|s_0 - a_n\| \]
\[ = \|tx - z_n\| + \|B(t\hat{x}) - B\hat{a}_n\| \leq 2\delta_2 \leq \delta_1, \]
(use (14)). Moreover, we have (use (15))
\[
\frac{1}{2}\|v_n\|^2 + \frac{1}{2}\|u_n\|^2 - \left\|\frac{u_n + v_n}{2}\right\|^2
= \frac{1}{2}\|z_n - P_{\hat{a}_n}(z_n)\|^2 + \frac{1}{2}\|x'_n - P_{\hat{a}_n}(x'_n)\|^2 - \left\|\frac{x'_n + z_n}{2} - P_{\hat{a}_n}\left(\frac{x'_n + z_n}{2}\right)\right\|^2
\leq \frac{1}{2}\|tx - P_{\hat{a}_n}(tx)\|^2 + \frac{1}{2}\|tx_n - P_{\hat{a}_n}(tx_n)\|^2 - \left\|\frac{tx + tx_n}{2} - P_{\hat{a}_n}\left(\frac{tx + tx_n}{2}\right)\right\|^2
+ \frac{1}{2}\|z_n - tx - P_{\hat{a}_n}(z_n - tx)\|\left(\|z_n - P_{\hat{a}_n}(z_n)\| + \|tx - P_{\hat{a}_n}(tx)\|\right)
+ \frac{1}{2}\|x'_n - tx_n - P_{\hat{a}_n}(x'_n - tx_n)\|\left(\|x'_n - P_{\hat{a}_n}(x'_n)\| + \|tx_n - P_{\hat{a}_n}(tx_n)\|\right)
+ \frac{1}{2}\left(\frac{z_n - tx}{2} - P_{\hat{a}_n}\left(\frac{z_n - tx}{2}\right)\right) + \left\|\frac{x'_n - tx_n}{2} - P_{\hat{a}_n}\left(\frac{x'_n - tx_n}{2}\right)\right\|
\cdot \left(\|z_n - P_{\hat{a}_n}(z_n)\| + \|x'_n - P_{\hat{a}_n}(x'_n)\| + \|tx - P_{\hat{a}_n}(tx)\| + \|tx_n - P_{\hat{a}_n}(tx_n)\|\right)
\leq 10\delta_2^2 (tK + 1)(\|s_0\|\|s_0\|^{-1} + 2)^2 \leq \delta
\]
(use (6), (18) and (19)). Therefore, by (7),
\[ \frac{\varepsilon}{2} \geq \|u_n - v_n\| = \|x'_n - P_{\hat{a}_n}x'_n - z_n + P_{\hat{a}_n}z_n\| = \|x'_n - z_n\|. \]

Thus by (18), (19) and (20), we have
\[ \|tx_n - tx\| \leq \|tx_n - x'_n\| + \|x'_n - z_n\| + \|z_n - tx\| \]
\[ \leq 2\delta_2 + \frac{\varepsilon t}{2} + \delta_2 = 3\delta_2 + \frac{\varepsilon t}{2} \leq \frac{3\varepsilon t}{8} + \frac{\varepsilon t}{2} < \varepsilon. \]
Thus $\|x_n - x\| < \varepsilon$, a contradiction and the proof is finished.
We end the paper with the following apparently open problem: Let $X$ be a Banach space and $Y$ be such a subspace of $X$ that both $Y$ and $X/Y$ admit equivalent strictly convex norms. Must $X$ admit an equivalent strictly convex norm?

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