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THE ATOMIC DECOMPOSITION OF BESOV-BERGMAN-LIPSCHITZ SPACES

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Dedicated to Professor Daniel Waterman

ABSTRACT. Let b denote a special atom, $b\colon [-\pi,\pi) \to R$, $b(t) = 1/2\pi$ or, for any interval I in $[-\pi,\pi)$, $b(t) = -|I|^{-1/p}\chi_R(t) + |I|^{-1/p}\chi_L(t)$, L is the left half of I, R is the right half, $|I|$ denotes the length of I and χ_E the characteristic function of E. For $1/2 < p < \infty$, let (b_n) be special atoms and (c_n) a sequence of real numbers; then we define the space

$$B^p = \left\{ f\colon [-\pi,\pi) \to R; f(t) = \sum_{n=1}^{\infty} c_n b_n(t), \sum_{n=1}^{\infty} |c_n| < \infty \right\}.$$

We endow B^p with the norm $\|f\|_{B^p} = \operatorname{Inf} \sum_{n=1}^{\infty} |c_n|$, where the infimum is taken over all possible representations of f.

In the early 1960s, the following spaces were introduced, now known as Besov-Bergman-Lipschitz spaces. For $0 < \alpha < 1$, $1 \leq r$, $s \leq \infty$, let

$$\Lambda(\alpha,r,s) = \left\{ f\colon [-\pi,\pi) \to R, \|f\|_{\Lambda(\alpha,r,s)} = \|f\|_r + \left(\int_{-\pi}^{\pi} \frac{(\|f(x+t)-f(x)\|_r)^s}{|t|^{1+\alpha s}}\, dt \right)^{1/s} < \infty \right\}$$

where $\| \ \|_r$ is the Lebesgue space L^r-norm.

Now we write down the main theorem of this paper which is as follows.

THEOREM C. *$f \in B^p$ for $1 < p < \infty$ if and only if $f \in \Lambda(1-1/p,1,1)$. Moreover, there are absolute constants M and N such that*

$$N\|f\|_{B^p} \leq \|f\|_{\Lambda(1-1/p,1,1)} \leq M\|f\|_{B^p}.$$

Let b denote a special atom, $b\colon [-\pi,\pi) \to R$, $b(t) = 1/2\pi$ or, for any interval I in $[-\pi,\pi)$, $b(t) = -|I|^{-1/p}\chi_R(t) + |I|^{-1/p}\chi_L(t)$, L is the left half of I, R is the right half, $|I|$ denotes the length of I and χ_E the characteristic function of E. For $1/2 < p < \infty$, let (b_n) be special atoms and (c_n) a sequence of real numbers; then we define the space

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We endow B^p with the norm $\|f\|_{B^p} = \operatorname{Inf} \sum_{n=1}^{\infty} |c_n|$, where the infimum is taken over all possible representations of f.

These spaces were originally introduced by the author who has extensively studied them. The reader is referred to [**3**–**11**].

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In the early 1960s the following spaces were introduced, now known as Besov-Bergman-Lipschitz spaces. For $0 < \alpha < 1$, $1 \le r$, $s \le \infty$, let

$$\Lambda(\alpha, r, s) = \left\{ f\colon [-\pi,\pi) \to R,\ \|f\|_{\Lambda(\alpha,r,s)} = \|f\|_r + \left(\int_{-\pi}^{\pi} \frac{(\|f(x+t)-f(x)\|_r)^s}{|t|^{1+\alpha s}}\,dt \right)^{1/s} < \infty \right\}$$

where $\|\ \|_r$ is the Lebesgue space L^r-norm.

These spaces have been studied in depth in [**13**, **14**, **15** and **17**].

We have the following result:

THEOREM A (EMBEDDING THEOREM). *If $f \in B^p$, $1 < p < \infty$, then $f \in \Lambda(1-1/p,1,1)$. Moreover, $\|f\|_{\Lambda(1-1/p,1,1)} \le C_p \|f\|_{B^p}$, where C_p is an absolute constant depending only on p.*

PROOF. First of all we notice that the operator $T_a f = f^a$, where $f^a(x) = f(x-a)$ maps B^p into B^p continuously, so that we just need to prove this result for

$$f_h(t) = \frac{-1}{(2h)^{1/p}} \chi_{[-h,0)}(t) + \frac{1}{(2h)^{1/p}} \chi_{[0,h]}(t), \qquad h > 0,$$

which will follow easily from the estimate for

$$g_h(t) = \frac{1}{h^{1/p}} \chi_{[0,h]}(t).$$

Therefore, all we need to prove is that

$$I_p = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{|g_h(x) - g_h(y)|}{|x-y|^{2-1/p}}\,dx\,dy \le C_p,$$

where C_p depends only on p.

In fact,

$$I_p \le \frac{2}{h^{1/p}} \int_0^h \int_h^\infty \frac{1}{|x-y|^{2-1/p}}\,dx\,dy \le \frac{2p^2}{p-1}.$$

Our argument shows that, for f_h as above, we have $\|f_h\|_{\Lambda(1-1/p,1,1)} < C_p$, and consequently, if $f \in B^p$ it follows that $\|f\|_{\Lambda(1-1/p,1,1)} \le C_p \|f\|_{B^p}$. So Theorem A is proved.

Because of Theorem A we may regard B^p as a subset of $\Lambda(1-1/p,1,1)$, so that we have

LEMMA B. *B^p is a dense subset of* $\Lambda(1-1/p,1,1)$.

Before we prove this lemma we would like to point out some definitions and results in order to make this presentation reasonably self-contained.

We define the Lipschitz space $\operatorname{Lip}\alpha = \{g\colon [-\pi,\pi) \to R$ continuous, such that $g(x+h) - g(x) = O(h^\alpha)\}$ for $0 < \alpha < 1$. The infimum of the constants in the definition of $\operatorname{Lip}\alpha$ is taken as $\|g\|_{\operatorname{Lip}\alpha}$. Also $\operatorname{Lip}'\alpha = \{g'\colon g \in \operatorname{Lip}\alpha\}$, where the prime indicates differentiation. We endow $\operatorname{Lip}'\alpha$ with the norm $\|g'\|_{\operatorname{Lip}'\alpha} = \|g\|_{\operatorname{Lip}\alpha}$.

A result that the interested reader can see in [**7** and **9**] is that the dual of B^p is equivalent to $\mathrm{Lip}'1/p$ for $1 < p < \infty$. In the sense that if ψ is a bounded linear functional on B^p there is a unique g in $\mathrm{Lip}\,1/p$ such that

$$\psi(f) = \lim_{r\to 1} \int_{-\pi}^{\pi} f_r(t) g_r'(t)\,dt \quad \text{for any } f \text{ in } B^p,$$

where $g_r = P_r * g$ and $f_r = P_r * f$ is the Poisson integral of g (respectively f).

In [**13**] Flett showed that the dual space of $\Lambda(1 - 1/p, 1, 1)$ is equivalent to $\mathrm{Lip}'\,1/p$, by means of the same representation above.

PROOF OF LEMMA B. Clearly B^p is closed under translations and dilations. If B^p is not dense in $\Lambda(1 - 1/p, 1, 1)$, then there exist a $g \in \mathrm{Lip}\,1/p$ such that $\|g\|_{\mathrm{Lip}1/p} = 1$ and $\lim_{r\to 1} \int_\pi^\pi f_r(t) g_r'(t)\,dt = 0$, for any $f \in B^p$, where g_r and f_r are as above. But this must hold for any translation or dilation of f in B^p. This implies g is constant and $\|g\|_{\mathrm{Lip}1/p} = 0$, which is absurd. Hence the lemma is proved.

Notice that we have the following situation, the spaces B^p and $\Lambda(1 - 1/p, 1, 1)$ have the same duals; see comments right after Lemma B. Moreover Theorem A tells us that B^p is regarded as a dense subset of $\Lambda(1 - 1/p, 1, 1)$, so that the classic theorem in functional analysis ensures us that B^p and $\Lambda(1 - 1/p, 1, 1)$ are equivalent as Banach spaces; for instance see Theorem 4.14 of [**16**, p. 96] and Corollary 2.12c of [**16**, p. 48]. Really what we have proved is that the embedding operator $A\colon B^p \to \Lambda(1 - 1/p, 1, 1)$ defined by $A(f) = f$ is a Banach space isomorphism.

Now we write down the main theorem of this paper which is as follows:

THEOREM C. *$f \in B^p$ for $1 < p < \infty$ if and only if $f \in \Lambda(1 - 1/p, 1, 1)$. Moreover, there are absolute constants M and N such that*

$$N\|f\|_{B^p} \le \|f\|_{\Lambda(1-1/p,1,1)} \le M\|f\|_{B^p}.$$

REMARK. The proof of Theorem A should neither require cancellation nor the special form of a special atom. This indicates that if we consider a function supported on an interval $I \subseteq [-\pi, \pi)$ and $|a(x)| \le |I|^{-1/p}$ then $a \in \Lambda(1 - 1/p, 1, 1)$ and $\|a\|_{\Lambda(1-1/p,1,1)} \le M < \infty$ where M is an absolute constant which depends only on p. Therefore, we can define the space

$$A^p = \left\{ f\colon [-\pi, \pi) \to R, f(t) = \sum_{n=1}^{\infty} c_n a_n(t), \sum_{n=1}^{\infty} |c_n| < \infty \right\},$$

where a_k is supported on an interval I_k and $|a_k(t)| \le |I_k|^{-1/p}$. We endow A^p with the norm $\|f\|_{A^p} = \mathrm{Inf} \sum_{n=1}^{\infty} c_n a_n(t)$, where the infimum is taken over all possible representations of f. Therefore, we have

$$B^p \subseteq A^p \subseteq \Lambda(1 - 1/p, 1, 1),$$

and consequently, B^p is equivalent to A^p as Banach spaces with equivalent norms.

We would like to point out that the proof that A^p is continuously contained in $\Lambda(1 - 1/p, 1, 1)$ was done independently by Mitchell Taibleson [**18**] in a personal letter sent to me. Also in the same letter he pointed out that we could also have molecular representations of these spaces.

We recall that one of our earlier results done jointly with Richard O'Neil and G. Sampson (see [**9**]) states that B^p is equivalent with the spaces J^p of all analytic functions in the disk for which

$$\|F\|_{J^p} = |F(0)| + \frac{1}{\pi}\int_0^1 \int_{-\pi}^{\pi} |F'(re^{i\theta})|(1-r)^{1/p-1}\,d\theta\,dr < \infty,$$

in the sense that $f \in B^p$ if and only if $F \in J^p$; moreover, the norms are equivalents, and $\lim_{r\to 1} \operatorname{Re} F(re^{i\theta}) = f(\theta)$ a.e., where the prime indicates differentiation.

The raison d'etre of this paper is that the equivalence of B^p and J^p, as was done in [**9**], carries a very tedious calculation, which makes it very uncomfortable and hard to follow.

The above equivalence shows that B^p can be identified with the boundary values of J^p, or we may regard it as an atomic decomposition of J^p, a real characterization.

This result, together with Theorem C, yields that J^p is equivalent to $\Lambda(1-1/p,1,1)$, which was done by M. Tailbleson in [**17**, p. 421].

For $p=1$ and other p we refer the reader to [**4**, **5** and **6**].

We believe that Theorem C will help us to have a better understanding of these Besov-Bergman-Lipschitz spaces.

Finally, we mention that the same technique used to prove Theorem C can be easily used to show that

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{|f(x)-f(y)|}{|x-y|}\,dx\,dy < \infty \quad \text{for any } f \in B^p.$$

A consequence of the boundedness of this integral is that any f in B^p satisfies Dini's condition and therefore the almost everywhere convergence is readily established (see [**11**]).

We would like to thank Professor Mitchell Taibelson with whom we had several conversations about these spaces B^p.

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