THE HUREWICZ IMAGE OF RAY'S ELEMENTS IN $MSP_*$

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Abstract. In this paper we determine an explicit formula for the symplectic Hurewicz homomorphism of Nigel Ray's sequence of torsion elements in the symplectic bordism ring.

Nigel Ray [7] defined a sequence of elements $\theta_n \in \pi_{4n-3} MSP$ of order two. Ray conjectured that the $\theta_{2n-1}$, $n \geq 2$, are zero, and Fred Roush has an unpublished and unavailable proof of this conjecture. On the other hand, Ray proved that $\phi_0 = \theta_1$ and the $\phi_n = \theta_{2n}$, $n \geq 1$, are all nonzero. This sequence of elements has played a central role in all recent computations involving the torsion elements of $\pi_\ast MSP$ either to describe them [3, 4, 6] or to eliminate them [8]. The underlying reason is that all torsion elements of $\pi_\ast MSP$ can be organized into sequences called families which are constructed from the $\phi_n$ using Massey products. The details can be found in [4, §9]. Ray proved that $\{\theta_n | n \geq 1\}$ is closed under the Landweber-Novikov operations. From [4, §9] it follows that each family is closed under the Landweber-Novikov operations, at least modulo the Adams filtration. This gives rise to the following inductive procedure for computing differentials in the Adams spectral sequence for $\pi_\ast MSP$ [3] and in the Atiyah-Hirzebruch spectral sequences for $\pi^2 MSP(0)$ [2] and $\pi^2 MSP$ [6]. The Landweber-Novikov operations induce degree lowering operations on these spectral sequences which commute with the differentials. Thus, if one inducts on the degree of members $X$ of a fixed family, it often happens that $d_r(X)$ is determined from the knowledge of $S_E d_r(X)$ for Landweber-Novikov operations $S_E$. By the construction of families in [4, §9] it follows that if we knew how the $S_E$ act on the $\phi_n$ then we would know how the $S_E$ act on all families. In summary, current methods of computing with $\pi_\ast MSP$ rely on ad hoc calculations of $S_E(\phi_n)$ for special cases of $E$.

In this paper we determine explicit formulas for $S_E(\theta_n)$ for all Landweber-Novikov operations $S_E$. Equivalently, we determined $h(\theta_n)$, where $h: \pi_\ast MSP \to MSP_\ast MSP$ is the Hurewicz homomorphism. Recall [1] that $MSP_\ast MSP(1)$ has a canonical $\pi_\ast MSP$-basis $\{1, b_0, b_1, \ldots, b_n, \ldots\}$, $\deg b_n = 4n$. Then the equation $h(x) = \Sigma E x_E b_E$ is equivalent to saying that the Landweber-Novikov operations on $x$ are given by $S_E(x) = x_E$ for all $E$. Here $E = (e_1, \ldots, e_n)$, $b_E = b_1^{e_1} \cdots b_n^{e_n}$ and $x, x_E$ are in $\pi_\ast MSP$.

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Recall that Kaoru Morisugi [6] proved the following recursive formula for $h(\theta_n)$:

$$
\sum_{k=1}^{n-1} B_{n-k}^k h(\theta_k) + h(\theta_n) = \sum_{j=1}^{n} (n - j + 1) b_{n-j} \theta_j,
$$

where $B = 1 + b_1 + \cdots + b_i + \cdots$ and $B_{n-k}^k$ denotes the component of $B^k$ of degree $4n - 4k$. Interpret (1) as a sequence of linear equations with unknowns $h(\theta_k)$ and constant terms

$$
\sum_{j=1}^{n} (n - j + 1) b_{n-j} \theta_j.
$$

The coefficient matrix $C$ is lower triangular with ones on the diagonal and has $(i, j)$-entry $C_{ij} = B_{i-j}^j$ for $i > j$. We digress briefly to study $C$. Recall [1] that $\text{MSp} \star \text{MSp}$ is a Hopf algebra with coproduct $\Delta$ given by

$$
\Delta(B) = \sum_{t=0}^{\infty} B^{t+1} \otimes b_t.
$$

Thus for $i > j$,

$$
\Delta(C_{ij}) = \Delta(B)^{i-j}_j = \left( \sum_{t=0}^{\infty} B^{t+1} \otimes b_t \right)^{i-j}
$$

$$
= \left( \sum_{s_1, \ldots, s_j \geq 0} B_1^{s_1} \cdots B_j^{s_j} \otimes b_{s_1} \cdots b_{s_j} \right)^{i-j}
$$

$$
= \sum_{p=0}^{i-j} B_{i-j-p}^{s+p} \otimes B_p^j = \sum_{q=j}^{i} B_{q-j}^{s+q} \otimes B_{q-j}^j \quad \text{where} \quad q = p + j
$$

$$
= \sum_{q=j}^{i} C_{iq} \otimes C_{qj}.
$$

Thus by [5, Lemma 2.2], Cramer's rule applies to the system of linear equations (1) to determine the unique solution as

$$
h(\theta_n) = \sum_{j=1}^{n} \left[ (n - j + 1) b_{n-j} + \sum_{k=0}^{n-j-1} (k + 1) b_k \chi(B) B_{n-j-k}^{j+k} \right] \theta_j.
$$

In this formula $\chi$ is the conjugation in the Hopf algebra $\text{MSp} \star \text{MSp}$. By [5, Lemma 2.3],

$$
\chi(B)^{i-j}_n = \sum_{r \geq 0} \sum_{n > q_r > \cdots > q_1 > t} (-1)^{r+1} B_{n-q_r}^{q_r} B_{q_{r-1}}^{q_{r-1}} \cdots B_{q_1}^{q_1} B_{q_1-t}^i.
$$

Taking $n = 2m$ in formula (2) we obtain

$$
h(\phi_m) = \sum_{i=1}^{m} \left[ b_{2m-2i} + \sum_{h=0}^{m-i-1} b_{2h} \chi(B) B_{2m-2h-2i}^{2h+2i} \right] \phi_i
$$

$$
+ \sum_{j=1}^{m} \left[ \sum_{k=0}^{m-j} b_{2k} \chi(B) B_{2m-2j-2k+1}^{2j+2k-1} \right] \phi_j.
$$
By Roush’s unpublished result that \( \theta_{2j-1} = 0 \) for \( j \geq 2 \),

\[
\begin{align*}
(4)' \quad h(\phi_m) &= \sum_{i=1}^{m} \left[ b_{2m-2i} + \sum_{h=0}^{m-i-1} b_{2h} \chi(B)^{2h+2i} \right] \phi_i \\
&+ \sum_{k=0}^{m-1} b_{2k} \chi(B)^{2k+1} \phi_0.
\end{align*}
\]

On the other hand, one could take \( n = 2m - 1 \) in (2) to obtain

\[
(5) \quad h(\theta_{2m-1}) = \sum_{i=1}^{m} \left[ b_{2m-2i} + \sum_{h=0}^{m-i-1} b_{2h} \chi(B)^{2h+2i-1} \right] \theta_{2i-1}.
\]

Note that we exclude the term \( i = 1 \) in this sum because

\[
b_{2m-2} + \sum_{h=0}^{m-2} b_{2h} \chi(B)^{2h+1} = 0.
\]

This is a consequence of the definition of \( \chi \) and

\[
\Delta(b_{2m-2}) = 1 \otimes b_{2m-2} + \sum_{h=0}^{m-2} B_{2m-2h-2} \otimes b_{2h}.
\]

Morisugi [6, Proposition 4.3] observed that the formula for \( h(\theta_{2m-1}) \) would not contain \( \theta_1 \) nor \( \theta_{2n}, n \geq 1 \). He denotes the coefficient of \( \theta_i \) in \( h(\theta_n) \) by \( f_i(n) \). In this notation we can rewrite (2) as

\[
(6) \quad f_i(n) = (n - i + 1) b_{n-i} + \sum_{k=0}^{n-i-1} (k + 1) b_k \chi(B)^{n-i-k}.
\]

BIBLIOGRAPHY


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