THE HUREWICZ IMAGE OF RAY'S ELEMENTS IN $\text{MSp}_*$

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Abstract. In this paper we determine an explicit formula for the symplectic Hurewicz homomorphism of Nigel Ray's sequence of torsion elements in the symplectic bordism ring.

Nigel Ray [7] defined a sequence of elements $\theta_n \in \pi_{4n-3}\text{MSp}$ of order two. Ray conjectured that the $\theta_{2n-1}$, $n \geq 2$, are zero, and Fred Roush has an unpublished and unavailable proof of this conjecture. On the other hand, Ray proved that $\phi_n = \theta_{2n}$, $n \geq 1$, are all nonzero. This sequence of elements has played a central role in all recent computations involving the torsion elements of $\pi_*\text{MSp}$ either to describe them [3, 4, 6] or to eliminate them [8]. The underlying reason is that all torsion elements of $\pi_*\text{MSp}$ can be organized into sequences called families which are constructed from the $\phi_n$ using Massey products. The details can be found in [4, §9]. Ray proved that $\{\theta_n|n \geq 1\}$ is closed under the Landweber-Novikov operations. From [4, §9] it follows that each family is closed under the Landweber-Novikov operations, at least modulo the Adams filtration. This gives rise to the following inductive procedure for computing differentials in the Adams spectral sequence for $\pi_*\text{MSp}$ [3] and in the Atiyah-Hirzebruch spectral sequences for $\pi^\ast_*\text{MSp}(n)$ [2] and $\pi^\ast_*\text{MSp}$ [6]. The Landweber-Novikov operations induce degree lowering operations on these spectral sequences which commute with the differentials. Thus, if one inducts on the degree of members $X$ of a fixed family, it often happens that $d_r(X)$ is determined from the knowledge of $S_E d_r(X)$ for Landweber-Novikov operations $S_E$. By the construction of families in [4, §9] it follows that if we knew how the $S_E$ act on the $\phi_n$ then we would know how the $S_E$ act on all families. In summary, current methods of computing with $\pi_*\text{MSp}$ rely on ad hoc calculations of $S_E(\phi_n)$ for special cases of $E$.

In this paper we determine explicit formulas for $S_E(\theta_n)$ for all Landweber-Novikov operations $S_E$. Equivalently, we determined $h(\theta_n)$, where $h: \pi_*\text{MSp} \to \text{MSp}_*\text{MSp}$ is the Hurewicz homomorphism. Recall [1] that $\text{MSp}_*\text{MSp}(1)$ has a canonical $\pi_*\text{MSp}$-basis $\{1, b_0, b_1, \ldots, b_n, \ldots\}$, $\deg b_n = 4n$. Then the equation $h(x) = \sum E x_E b_E$ is equivalent to saying that the Landweber-Novikov operations on $x$ are given by $S_E(x) = x_E$ for all $E$. Here $E = (e_1, \ldots, e_n)$, $b_E = b_1^{e_1} \cdots b_n^{e_n}$ and $x, x_E$ are in $\pi_*\text{MSp}$. 

Received by the editors June 27, 1984.

1980 Mathematics Subject Classification. Primary 55N22.

1 This research was partially supported by a grant from the Natural Sciences and Engineering Research Council of Canada.
Recall that Kaoru Morisugi [6] proved the following recursive formula for \( h(\theta_n) \):

\[
\sum_{k=1}^{n-1} B_{n-k}^k h(\theta_k) + h(\theta_n) = \sum_{j=1}^{n} (n-j+1)b_{n-j} \theta_j,
\]

where \( B = 1 + b_1 + \cdots + b_i + \cdots \) and \( B_{n-k}^k \) denotes the component of \( B^k \) of degree \( 4n - 4k \). Interpret (1) as a sequence of linear equations with unknowns \( h(\theta_k) \) and constant terms

\[
\sum_{j=1}^{n} (n-j+1)b_{n-j} \theta_j.
\]

The coefficient matrix \( C \) is lower triangular with ones on the diagonal and has \((i, j)\)-entry \( C_{ij} = B_{i-j}^j \) for \( i > j \). We digress briefly to study \( C \). Recall [1] that \( MSp \ast MSp \) is a Hopf algebra with coproduct \( \Delta \) given by

\[
\Delta(B) = \sum_{t=0}^{\infty} B^{t+1} \otimes b_t.
\]

Thus for \( i > j \),

\[
\Delta(C_{ij}) = \Delta(B)^i_{i-j} = \left( \sum_{t=0}^{\infty} B^{t+1} \otimes b_t \right)^{i-j}
\]

\[
= \left( \sum_{s_1, \ldots, s_j \geq 0} B^{s_1 + \cdots + s_j + j} \otimes b_{s_1} \cdots b_{s_j} \right)^{i-j}
\]

\[
= \sum_{p=0}^{i-j} B^{i-j+p} \otimes B^j_p = \sum_{q=j}^{i} B^{q} \otimes B_{q-j}^j \quad \text{where} \quad q = p + j
\]

\[
= \sum_{q=j}^{i} C_{iq} \otimes C_{qj}.
\]

Thus by [5, Lemma 2.2], Cramer's rule applies to the system of linear equations (1) to determine the unique solution as

\[
h(\theta_n) = \sum_{j=1}^{n} \left[ (n-j+1)b_{n-j} + \sum_{k=0}^{n-j-1} (k+1)b_k \chi(B)_{n-j-k}^{i+k} \right] \theta_j.
\]

In this formula \( \chi \) is the conjugation in the Hopf algebra \( MSp \ast MSp \). By [5, Lemma 2.3],

\[
\chi(B)^{i}_{n-\ell} = \sum_{r \geq 0} \sum_{n > q_1 > \cdots > q_r > \ell} (-1)^{r+1} B^{q_r} \cdots B^{q_2-q_1} B^t_{q_1-q_2-\ell}.
\]

Taking \( n = 2m \) in formula (2) we obtain

\[
h(\phi_m) = \sum_{i=1}^{m} b_{2m-2i} + \sum_{h=0}^{m-i-1} b_{2h} \chi(B)^{2h+2i}_{2m-2h-2i} \phi_i
\]

\[
+ \sum_{j=1}^{m-j} \sum_{k=0}^{m-j} b_{2k} \chi(B)^{2j+2k-1}_{2m-2j-2k+1} \theta_{2j-1}.
\]
By Roush's unpublished result that \( \theta_{2j-1} = 0 \) for \( j \geq 2 \),

\[
(4)'
\begin{align*}
\phi^j_m &= \sum_{i=1}^{m} \left[ b_{2m-2i} + \sum_{h=0}^{m-i-1} b_{2h} \chi(B)_{2m-2h-2i} \right] \phi_i \\
&+ \sum_{k=0}^{m-1} b_{2k} \chi(B)_{2m-2k-1} \phi_0.
\end{align*}
\]

On the other hand, one could take \( n = 2m - 1 \) in (2) to obtain

\[
(5)
\begin{align*}
\phi^j_{2m-1} &= \sum_{i=2}^{m} \left[ b_{2m-2i} + \sum_{h=0}^{m-i-1} b_{2h} \chi(B)_{2m-2h-2i} \right] \theta_{2i-1}.
\end{align*}
\]

Note that we exclude the term \( i = 1 \) in this sum because

\[
\sum_{h=0}^{m-2} b_{2h} \chi(B)_{2m-2h-2} = 0.
\]

This is a consequence of the definition of \( \chi \) and

\[
\Delta(b_{2m-2}) = 1 \otimes b_{2m-2} + \sum_{h=0}^{m-2} B_{2m-2h-2} \otimes b_{2h}.
\]

Morisugi [6, Proposition 4.3] observed that the formula for \( h(\theta_{2m-1}) \) would not contain \( \theta_1 \) nor \( \theta_{2n}, n \geq 1 \). He denotes the coefficient of \( \theta_i \) in \( h(\theta_n) \) by \( f_i(n) \). In this notation we can rewrite (2) as

\[
(6)
\begin{align*}
f_i(n) &= (n-i+1)b_{n-i} + \sum_{k=0}^{n-i-1} (k+1)b_k \chi(B)_{n-i-k}.
\end{align*}
\]

**BIBLIOGRAPHY**


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