MINIMUM EIGENVALUES FOR
POSITIVE, ROCKLAND OPERATORS

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Abstract. Let \( L \) be a positive, Rockland operator of homogeneous degree \( \gamma \). The minimum eigenvalue of \( d^n(L) \) increases as the \( \gamma \)th power of the homogeneous distance from the origin of the orbit corresponding to \( \pi \).

Let \( G \) be a connected, simply connected nilpotent Lie group with Lie algebra \( \mathfrak{g} \). Let \( A \) be a diagonalizable operator on \( \mathfrak{g} \) with eigenvalues \( 1 = \gamma_1 < \cdots < \gamma_p \), such that for each \( r > 0 \), the operator \( \delta_r : = r^A \) is an automorphism of \( \mathfrak{g} \) and, hence, determines an automorphism, again denoted by \( \delta_r \), of \( G \). Let \( | \cdot | \) denote a homogeneous gauge on \( G \); i.e., \( | \cdot | \) is a continuous, nonnegative function on \( G \) with \( |g| = 0 \) if, and only if, \( g = e \), and satisfying \( |\delta_r g| = r|g| \) for \( r > 0 \). Let \( \mathfrak{g}^* \) be the dual of \( \mathfrak{g} \), and let \( \delta_r^* \) denote the adjoint of \( \delta_r \) acting on \( \mathfrak{g}^* \). We define \( | \cdot | \) on \( \mathfrak{g} \) by \( |X| = |\exp X| \) and on \( \mathfrak{g}^* \) as follows: fix a basis \( \{ X_1, \ldots, X_d \} \) of \( \mathfrak{g} \) consisting of eigenvectors of \( A \), and let \( \{ X_1^*, \ldots, X_d^* \} \) be the dual basis of \( \mathfrak{g}^* \). For \( \xi \in \mathfrak{g}^* \) we set \( |\xi| = |\Sigma(\xi, X_\ell) X_\ell| \). One easily sees that \( |\delta_r X| = r|X| \) and \( |\delta_r^* \xi| = r|\xi| \) for \( X \in \mathfrak{g}, \xi \in \mathfrak{g}^* \), and \( r > 0 \).

Let \( g \to \text{Ad}^* g \) be the coadjoint representation of \( G \) on \( \mathfrak{g}^* \), and for \( \xi \in \mathfrak{g}^* \) let \( O(\xi) = \text{Ad}^* G \cdot \xi \). Set

\[
|O(\xi)| = \inf \{|\xi'| \mid \xi' \in O(\xi)\}.
\]

By Kirillov theory [1] the equivalence classes of irreducible unitary representations of \( G, \hat{G} \), can be identified with \( \{ O(\xi) \mid \xi \in \mathfrak{g}^* \} \). Given \( \xi \in \mathfrak{g}^* \), we denote by \( \pi_\xi \) the element of \( \hat{G} \) corresponding to \( O(\xi) \), and by \( d\pi_\xi \) the resulting representation of \( \mathfrak{U}(\mathfrak{g}) \), the universal enveloping algebra of \( \mathfrak{g} \).

An element \( L \) of \( \mathfrak{U}(\mathfrak{g}) \) is said to be homogeneous of degree \( \gamma \) if \( L(f \circ \delta_r) = r^\gamma(Lf) \circ \delta_r \) for all smooth functions of \( f \). As in [2], a homogeneous operator \( L \) is called a Rockland operator if \( d\pi_\xi(L) \) is injective (on the space of smooth vectors) for each \( \xi \neq 0 \). \( L \) is said to be positive if \( (Lf, f) \geq 0 \) for all \( f \in C_0 (G) \). The proof by Nelson and Steinspring [6] that \( d\pi(L) \) is essentially selfadjoint for any elliptic operator and any unitary representation \( \pi \) uses, in fact, only the hypoellipticity of \( L \). Thus, by the theorem of Helffer and Nourrigat [3] \( d\pi_\xi(L) \) is essentially selfadjoint for any Rockland operator. Therefore, if \( L \) is a positive, Rockland operator, both \( L \) and \( d\pi_\xi(L) \) are infinitesimal generators of contraction semigroups. In [2] Folland

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and Stein show that the densities of the semigroup generated by \( L, \{ P_t \}_{t>0} \), are Schwartz functions on \( G \). One can easily show that the infinitesimal generator of \( \{ \pi_\xi(P_t) \}_{t>0} \) is \( d\pi_\xi(L) \). Since \( \pi_\xi \), integrated to \( L^1(G) \), maps into compact operators, \( d\pi_\xi(L) \) has as eigenvalues a discrete subset \( \sigma_\xi(L) \subset (0, \infty) \) for each \( 0 \neq \xi \in g^* \).

**Theorem.** Let \( L \) be a positive, Rockland operator of degree \( \gamma \). There is a \( c > 0 \) such that, for all \( \xi \in g^* \), \( \min \{ \alpha \in \sigma_\xi(L) \} \geq c|O(\xi)|^{\gamma} \).

**Proof.** Define \( \delta_\star \) on \( g^* \) by \( (\delta_\star \xi, X) = (\xi, \delta_\star X) \). Then
\[
\alpha(\xi^\sharp L) \equiv \alpha(d\pi_\xi(L) \circ \delta_\star) = r^\gamma d\pi_\xi(L).
\]
Indeed, if \( \alpha \in \text{Aut}(G) \), we may also regard \( \alpha \) as an automorphism of \( g \) and similarly define \( \alpha^\star \in \text{End}(g^*) \). If \( \pi \in \hat{G} \) we set \( \pi^\alpha(x) = \pi(\alpha(x)) \). One can easily verify from Kirillov theory that \( \pi_\xi^\alpha = \pi_{\alpha^\star \xi} \). This implies (i) for \( \alpha = \delta_\star \) and \( L \) homogeneous of degree \( \gamma \).

From (i) one easily shows
\[
\min \{ \alpha \in \sigma_\delta \xi(L) \} \geq s^\gamma \min \{ \alpha \in \sigma_\xi(L) \}
\]
by merely noting that \( f \) is an eigenvector \( d\pi_{\delta \star \xi}(L) \) if, and only if, \( f \circ \delta_\star / s \) is an eigenvector for \( d\pi_\xi(\delta_\star L) = s^\gamma d\pi_\xi(L) \), and the corresponding eigenvalues differ by factor of \( s^\gamma \).

We now set \( B = \{ \xi \in g^* | |\xi| = |O(\xi)| \} \). It is clear that \( B \) is compact and for each \( \xi \in g^* \) there is a \( \xi' \in B \) such that \( O(\xi) = O(\delta^* \xi') \), where \( s = |O(\xi)| \). Also, one has
\[
\inf \{ \alpha | \alpha \in \sigma_\xi(L), \xi \in B \} > 0.
\]
To see this, use induction on \( k \), where \( B_k = \{ \xi \in B | \dim O(\xi) = 2k \} \). The case \( k = 0 \) is obvious, since for \( \xi \in B_0 \), \( \pi_\xi \) is a character on \( G \) and, thus, \( \sigma_\xi(L) = \{ d\pi_\xi(L) \} \).

Assume \( \inf \{ \alpha | \alpha \in \sigma_\xi(L), \xi \in B_l, l < k \} \geq 0 \) and suppose \( \alpha_n \in \sigma_\xi(L), \xi_n \in B_k, \alpha_n \to 0 \), and \( \xi_n \to \xi_0 \in g^* \). As in [1, p. 105] pick bases \( \{ X^{(n)}_1, \ldots, X^{(n)}_d \} \) of \( g \) so that \( \{ X^{(n)}_1, \ldots, X^{(n)}_d \} \) is a basis for a polarization \( h_n \) of \( \xi_n \), so that \( X_i = \text{Lim} X^{(n)}_i \) exists for \( 1 \leq i \leq d \) and \( \{ X_1, \ldots, X_{d-k} \} \) is contained in a basis for a polarization of \( \xi_0 \). Define \( \beta_n : R^{d-k} \times R^k \to G \) by
\[
\beta_n(t, s) = \exp t_1 X^{(n)}_1 \cdots \exp s_k X^{(n)}_k,
\]
where \( t = (t_1, \ldots, t_{d-k}) \) and \( s = (s_1, \ldots, s_k) \). Define \( \tilde{\beta}_n : L^2(H_n \setminus G, \chi_{\xi_n}) \to L^2(R^k) \) by
\[
(\tilde{\beta}_n f)(s) = f(\beta_n(0, s)).
\]
With the correct normalization of Lebesgue measure on \( R^k \), \( \tilde{\beta}_n \) is an isometric isomorphism. Thus \( \pi_{\xi_n} \approx \tilde{\pi}_{\xi_n} = \tilde{\beta}_n \circ \pi_\xi \circ \tilde{\beta}_n^{-1} \). Also, one has that \( \tilde{\pi}(g) = \text{Lim} \tilde{\pi}_{\xi_n}(g) \) exists for each \( g \in G \).

Let \( \{ P_t \}_{t \geq 0} \) be the semigroup generated by \( L \), and set
\[
R_1 = \int_0^\infty e^{-t} P_t dt.
\]
Then \( R_1 = (I + L)^{-1} \) (cf. [7]), and the assumptions imply that \( \beta_n = 1/(1 + \alpha_n) \in \sigma_{\xi_n}(R_1) \), the spectrum of \( \pi_{\xi_n}(R_1) \). Thus, \( \sup \{ \beta \in \sigma_{\xi_n}(R_1) \} \geq 1 \). We will show that this is impossible.

In [5, Proposition 6.3], it is shown that, by passing to a subsequence if necessary, there are only finitely many orbits in \( g^* \), with dimension \( 2k \), that are limit points in \( \hat{G} \) of \( \{ O(\xi_n) \} \). Denote this set by \( \mathcal{A}_1 \). Let \( \mathcal{A}_2 \) be the lower-dimensional orbits that are limit points of \( \{ o(\xi_n) \} \), and let \( \mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \). Then, by [4],

\[
\lim_n \left\| \pi_{\xi_n}(R_1) \right\| = \sup_{\xi \in \mathcal{A}} \left\| \pi_\xi(R_1) \right\| = \sup_{\xi \in \mathcal{A}_1} \left\| \pi_\xi(R_1) \right\|.
\]

Thus,

\[
\lim_n \sup \{ \beta \in \sigma_{\xi_n}(R_1) \} = \max \left\{ \sup_{\mathcal{A}_1} \left\| \pi_\xi(R_1) \right\|, \sup_{\mathcal{A}_2} \left\| \pi_\xi(R_1) \right\| \right\} < 1,
\]

since \( \mathcal{A}_1 \) is finite and the orbits in \( \mathcal{A}_2 \) have dimension less than \( 2k \).

**References**


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