DENSE PERIODICITY ON THE INTERVAL

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Abstract. We give a description of those continuous functions on the interval for which the set of periodic points is dense.

The purpose of this paper is to describe those continuous functions \( f \) on the interval for which the set of periodic points is dense. While this description could have been given in terms of \( f \), it is simpler and more informative to describe \( f^2 \).

Suppose that \( I \) is an interval, \( f: I \to I \) is continuous, and that the set of periodic points of \( f \) is dense in \( I \). What follows is a rough description of the graph of \( f^2 \): on the diagonal there are "blocks of orbit density," intervals \( J_i \) such that \( f^2(J_i) = J_i \). For each \( i \), the restriction of \( f^2 \) to \( J_i \) has a point whose orbit is dense in \( J_i \). Furthermore, if \( x \in I - \bigcup J_i \), then \( f^2(x) = x \). From this description it follows that if each period of \( f \) is a power of 2, then \( f \) is a period 2 homeomorphism. See the corollary.

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Theorem. Suppose that $f: I \to I$ is continuous, and that the set of periodic points of $f$ is dense in $I$. Then there is a collection (perhaps finite or empty) \{ $J_1, J_2, \ldots$ \} of mutually disjoint closed subintervals of $I$ such that (i) $f^2(J_i) = J_i$, (ii) for each $i$, there is a point $x_i \in J_i$ such that \{ $f^{4n}(x_i) | n \geq 0$ \} is dense in $J_i$, and (iii) if $x \in I - \bigcup_{i=1}^{\infty} J_i$, then $f^2(x) = x$.

Definitions and terminology. Throughout, $I$ will denote a closed interval, and $f: I \to I$ will be a continuous function. If $n$ is a nonnegative integer, $f^n$ is the $n$-fold composition of $f$. If $x \in I$, the orbit of $x$, $O(x)$ is \{ $f^n(x) | n \geq 0$ \}. The statement that $x$ is periodic means that there is an integer $n$ such that $f^n(x) = x$, and the statement that $x$ has period $n$ means that $n$ is a positive integer, $f^n(x) = x$, and if $k$ is an integer, $1 \leq k < n$, then $f^k(x) \neq x$.

Associated with $f: I \to I$ is the inverse limit space $$(I, f) = \{(x_0, x_1, \ldots) | f(x_{i+1}) = x_i\}$$ with metric $$d((x_0, x_1, \ldots), (y_0, y_1, \ldots)) = \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{2^i}.$$ In [B-M] and [B-M2] we began a study of the dynamics of functions on the interval by analyzing $(I, f)$. The theorem in this paper is a continuation of this program.

$(I, f)$ is a compact, connected metric space, and is an example of what Bing [Bi] has called a snakelike continuum. We will denote elements of $(I, f)$ by subbarred letters, as $x = (x_0, x_1, \ldots)$. The projection maps $\Pi_n$, of $(I, f)$ onto $I$ given by $\Pi_n(x) = x_n$ are continuous. If $H$ is a subcontinuum (compact, connected subspace) of $(I, f)$ we will let $H_n$ denote $\Pi_n(H)$. Note that $H_n$ is a closed interval or point and that $f(H_n+1) = H_n$.

If $f: I \to I$, then $f$ induces a homeomorphism $\tilde{f}: (I, f) \to (I, f)$ by $\tilde{f}((x_0, x_1, \ldots)) = (f(x_0), x_0, x_1, \ldots)$. We will utilize the fact that the intersection of any collection of subcontinua of $(I, f)$ is a subcontinuum of $(I, f)$. See [Bi].

If $S$ is a continuum, the statement that $S$ is indecomposable means that $S$ is not the union of two of its proper subcontinua. If $S$ is indecomposable and $H$ is a subcontinuum of $S$, then $H$ contains no open set in $S$ [H-Y, pp. 139–141].

Finally, in Lemma 2, we will utilize the following important construction due to Bing [Bi]. Suppose that $(I, f)$ contains no indecomposable subcontinuum with interior. For each $\bar{x} \in (I, f)$ let $g_{\bar{x}}$ be the intersection of all subcontinua of $(I, f)$ that contain interiorly a subcontinuum that contains $\bar{x}$ in its interior. Then $g_{\bar{x}}$ is a subcontinuum of $(I, f)$. Furthermore, the sets $g_{\bar{x}}$ partition $(I, f)$, and if we let $G = \{ g_{\bar{x}} | \bar{x} \in (I, f) \}$ with the quotient topology, then $G$ is an arc (i.e. homeomorphic with $I$). Moreover, $f$ induces a homeomorphism $\tilde{f}$ of $G$ onto $G$ given by $\tilde{f}(g_{\bar{x}}) = g_{\tilde{f}(\bar{x})}$. Bing also shows that $g_{\bar{x}}$ does not have interior.

Lemma 1. Suppose that $g: I \to I$ is continuous, that $(I, g)$ is indecomposable, and that the periodic points of $g$ are dense in $I$. Then $g^2$ has a dense orbit.
Proof. We will first argue that $g$ has a dense orbit. Suppose that $J$ is a subinterval of $I$ and that $x \in \text{int } I$. We will show that there is an integer $m$ such that $x \in g^m(J)$. Let $p$ be a periodic point in $\text{int } J$. Suppose that $p$ has period $k$. Let $L = \text{cl}(\bigcup_{n=0}^{\infty} g^n(J))$. Suppose that $L \neq I$, and that there is a point $y \in I - L$ such that $g^k(y) \subset L$. Then there is a periodic point $q \in I - L$ such that $g^k(q) \subset L$. Suppose that the period of $q$ is $j$. Then $g^j(q) = q$, but $g^k(q) \subset L$. This is a contradiction, and so $g^{-k}(L) = L$. Then we have $g^k(L) = L$, and $g^{-k}(L) = L$. Now, $(I, g)$ is homeomorphic with $(I, g^k)$, but since $g^{-k}(L) = L$, we have that $(L, g^k)$ is a proper subcontinuum of $(I, g^k)$ with interior. This contradicts the indecomposability of $(I, g)$ [H-Y, pp. 139–141]. Thus,

$$L = \text{cl}\left(\bigcup_{n=0}^{\infty} g^n(J)\right) = I.$$ 

It follows that there is an integer $m$ such that $x \in g^m(J)$. This argument shows that if $x \in \text{int } I$, then $\bigcup\{g^{-n}(x) | n \geq 0\}$ is dense in $I$. It follows that if $U$ is an open set in $I$, then $\bigcup_{n=0}^{\infty} g^{-n}(U) | n \geq 0\}$ is dense in $I$. Then using Lemma 3 of [A-Y] we have that $g$ has a dense orbit.

Now, since $g$ has a dense orbit, and $(I, g)$ is indecomposable, it follows from Theorem 3 of [B-M] that $g^2$ has a dense orbit.

Lemma 2. Suppose that the periodic points of $f$ are dense, and that $(I, f)$ contains no indecomposable subcontinua with interior. Then $f^2$ is the identity.

Proof. Since $(I, f)$ contains no indecomposable subcontinua with interior we may use Bing’s construction [Bi]. That is, there is an upper semicontinuous decomposition space $G$ of $(I, f)$ into subcontinua and points such that (1) the decomposition space $G$ is an arc, and (2) no element of $G$ has interior. Furthermore, the homeomorphism $\hat{f}: (I, f) \to (I, f)$ induces a homeomorphism $\hat{f}: G \to G$, by $\hat{f}(g_x) = g_{f(x)}$.

Since the periodic points of $f$ are dense, we have that the periodic points of $\hat{f}$ are dense, and that the periodic points of $\hat{f}$ are dense. Since $G$ is an arc and $\hat{f}$ is a homeomorphism, it follows that $\hat{f}^2$ is the identity.

We will next argue that every element in the decomposition $G$ is a point. Suppose that $H$ is a subcontinuum of $(I, f)$ and $H \in G$. Then $\hat{f}^2(H) = H$, and if $H_0 = \Pi_0(H)$, then $f^2(H_0) = f^2(\Pi_0(H)) = \Pi_0(f^2(H)) = \Pi_0(H) = H_0$. Thus, $H_0$ is invariant under $f^2$. Furthermore, if $K \in G$, $K \neq H$, then $f^2(K_0) = K_0$, and so $H_0 \cap K_0 = \emptyset$. Then if $H \in G$ we have that $f^{-2}(H_0) = H_0$. Then $H = \{(x_0, x_1, \ldots) | x_{2n} \in H_0\}$. Now if $\text{int } H_0 \neq \emptyset$, then $H$ has interior. This is a contradiction, and so $H_0$ is a point.

Now let $x \in I$, and let $\bar{x}$ be a point of $(I, f)$ with $\Pi_0(\bar{x}) = x_0 = x$. Let $g_x$ be the element of $G$ containing $\bar{x}$. Then from the previous discussion we have that $g_x = \{\bar{x}\}$ and that $\hat{f}^2(g_x) = g_x$. Thus, $\hat{f}^2(\bar{x}) = \bar{x}$, and so $f^2(x) = x$. This establishes Lemma 2.
Proof of Theorem. We suppose that the periodic points of $f$ are dense. Suppose that $H$ is an indecomposable subcontinuum of $(I, f)$ with interior. Since the periodic points of $f$ are dense, it follows that the periodic points of $\hat{f}$ are dense, and so there is a point $x \in \text{int } H$ which is periodic. Suppose that $x$ has period $k$.

Consider $H \cap \hat{f}^k(H)$. This is a subcontinuum of $H$ with interior, and so $\hat{f}^k(H) = H$. Then we have that $f^k(H_0) = f^k(\Pi_0(H)) = H_0$,

$$f^k(H_0) = H_0.$$ Consider the intervals $H_0, H_1, \ldots, H_{k-1}$. We have that $f(H_i) = H_{i-1} \pmod k$.

Suppose that $i \neq j$ and that $\text{int } H_i \cap \text{int } H_j \neq \emptyset$. We have then that $\hat{f}^i(H) \cap \hat{f}^j(H)$ is a subcontinuum of $\hat{f}^i(H)$ with interior. Since $\hat{f}^i(H)$ is indecomposable, we have that $\hat{f}^i(H) = \hat{f}^j(H)$ and so $H_i = H_j$. Thus the distinct intervals in $\{H_0, H_1, \ldots, H_{k-1}\}$ have disjoint interiors.

We will next show that there are at most two distinct intervals in the collection $\{H_0, H_1, \ldots, H_{k-1}\}$. Suppose the collection contains more than two distinct intervals. A consideration of cases reveals that there are integers $i, j, r, s$ such that $H_i, H_j, H_r, H_s$ are distinct, no interval in the collection lies between $H_i$ and $H_j$, and $H_r$ lies between $f^r(H_i)$ and $f^s(H_s)$. It then follows, since periodic points are dense and $\text{int } H_i \neq \emptyset$, that there is a periodic point $x$ between $H_i$ and $H_j$ such that $f(x) \in H_r$. That is, $x \notin \bigcup_{i=0}^{k-1} H_i$, $f^r(x) \in \bigcup_{i=0}^{k-1} H_i$, and $f^p(x) = x$ for some $p > 0$. But this is impossible since $f(\bigcup_{i=0}^{k-1} H_i) = \bigcup_{i=0}^{k-1} H_i$. Thus, there are at most two distinct intervals in the collection $\{H_0, H_1, \ldots, H_{k-1}\}$, and we have $f^2(H_i) = H_i$ for all $i \geq 0$.

Now let $\{H^1, H^2, H^3, \ldots\}$ be the collection of all indecomposable subcontinua of $(I, f)$ with interior. For each $i$, let $J_i = \Pi_0(H^i)$. It follows, as before, that $\text{int } J_i \cap \text{int } J_m = \emptyset$ if $i \neq m$.

From the previous argument we see that $f^2(J_i) = J_i$. Applying Lemma 1 to $f^2[J_i; J_i \rightarrow J_i]$, we see that there is a point $x_i \in J_i$ such that $(f^4n(x_i)|n \geq 0)$ is dense in $J_i$.

Let $J = \text{cl}(\bigcup_{i=1}^\infty J_i)$. Since $f^2(J_i) = J_i$ for all $i \geq 1$, it is clear that if $x \in J - \bigcup_{i=1}^\infty J_i$, then $f^2(x) = x$. Thus the theorem is established in the case that $J = I$.

Suppose $J \neq I$. Let $\{K_\alpha\}$ be the collection of all components of $J$. For each $\alpha$, $K_\alpha$ is a point or a closed interval. Now let $L$ be the decomposition space whose elements are the points in $L - J$ together with the $K_\alpha$. That is, $L = \{[x]|x \in I\}$ where $[x] = [y]$ if and only if $x = y$ or $x, y \in K_\alpha$ for some $\alpha$. It is clear that $L$, with the quotient topology, is an arc.

Let $P: I \rightarrow L$ be the projection map, $P(x) = [x]$. Then $P$ is a continuous and closed surjection. Furthermore, notice that $P$ is monotone, i.e. the inverse image of a continuum in $L$ is a continuum in $I$. Since $f$ respects the decomposition there is a uniquely defined continuous surjection $g$: $L \rightarrow L$ defined by $g([x]) = [f(x)]$, such that $g \circ P = P \circ f$. Then $P$ induces a continuous surjection on inverse limit spaces $\tilde{P}$: $(I, f) \rightarrow (L, g)$ by $\tilde{P}((x_0, x_1, \ldots)) = (P(x_0), P(x_1), \ldots)$.

We will now show that $(L, g)$ contains no indecomposable subcontinua with interior. Suppose, to the contrary, that $N$ is an indecomposable subcontinuum of $(L, g)$ that has nonempty interior. Then, since $P$ is monotone, it follows that $\tilde{P}$ is monotone, and hence $\tilde{P}^{-1}(N)$ is a subcontinuum of $(I, f)$. It follows from Zorn's Lemma that there is a minimal subcontinuum $M$ of $(I, f)$ such that $P(M) = N$. 
If $M$ decomposes as $M = M^1 \cup M^2$, $M^1$ and $M^2$ subcontinua of $M$, then $N = \overline{P}(M^1) \cup \overline{P}(M^2)$. Since $N$ is indecomposable, $\overline{P}(M^1) = N$ or $\overline{P}(M^2) = N$. But then, since $M$ is minimal, $M^1 = M$ or $M^2 = M$. Thus $M$ is indecomposable. We will show that $M$ has interior.

Let $C = \{x \in (L, g)|x_n \neq P(J) \text{ for all } n \geq 0\}$. Then, since $P(J)$ is closed and has empty interior, $C$ is a closed subset of $(L, g)$ with empty interior. Thus, $\operatorname{int} N \cap ((L, g) - C)$ is a nonempty open subset of $(L, g)$. Each point in $L - P(J)$ has a unique preimage under $P$ and it follows that each point in $(L, g) - C$ has a unique preimage under $\overline{P}$. Thus $\overline{P}^{-1}(\operatorname{int} N \cap ((L, g) - C))$ is an open subset of $M$.

We now have that $M$ is an indecomposable subcontinuum of $(I, f)$ with interior. Thus $M$ must be one of the $H^i$. But then $\overline{P}(M) = N$ is a single point and this contradicts the choice of $N$. Thus, $(L, g)$ has no indecomposable subcontinua with interior.

Since the periodic points under $f$ are dense in $I$, the periodic points under $g$ are dense in $L$. An application of Lemma 2 shows that $g^2$ is the identity on $L$. Thus $f^2$ is the identity on $I - \bigcup_{i=1}^\infty J_i$. This completes the proof of the theorem.

**Corollary.** Suppose that the periodic points of $f$ are dense, and that every period of $f$ is a power of 2. Then $f$ is a homeomorphism, and if $x \in I$, then $f^2(x) = x$. In particular, $f$ has no points of period other than 1 or 2.

**Proof.** Suppose that there is a subinterval $J$ of $I$ such that $f^2(J) = J$, and a point $x \in J$ such that $\{f^{4n}(x)|n \geq 0\}$ is dense in $J$. Then if $g = f^2|J$ we have that $g^2$ has a dense orbit on $J$. It follows from Theorem 13 of [B-M2] that $g$ has a point of odd period. Then, there is a point $x \in J$ which is periodic under $f$, and whose period under $f$ is not a power of 2. This is a contradiction, and so the collection $\{J_1, J_2, \ldots\}$ in the conclusion of the theorem is empty. It follows then, that if $x \in I$, $f^2(x) = x$. This establishes the corollary.