DENSE PERIODICITY ON THE INTERVAL

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Abstract. We give a description of those continuous functions on the interval for which the set of periodic points is dense.

The purpose of this paper is to describe those continuous functions \( f \) on the interval for which the set of periodic points is dense. While this description could have been given in terms of \( f \), it is simpler and more informative to describe \( f^2 \).

Suppose that \( I \) is an interval, \( f: I \to I \) is continuous, and that the set of periodic points of \( f \) is dense in \( I \). What follows is a rough description of the graph of \( f^2 \): on the diagonal there are “blocks of orbit density,” intervals \( J_i \) such that \( f^2(J_i) = J_i \). For each \( i \), the restriction of \( f^2 \) to \( J_i \) has a point whose orbit is dense in \( J_i \). Furthermore, if \( x \in I - \bigcup_i J_i \), then \( f^2(x) = x \). From this description it follows that if each period of \( f \) is a power of 2, then \( f \) is a period 2 homeomorphism. See the corollary.
Theorem. Suppose that \( f : I \to I \) is continuous, and that the set of periodic points of \( f \) is dense in \( I \). Then there is a collection (perhaps finite or empty) \( \{ J_1, J_2, \ldots \} \) of mutually disjoint closed subintervals of \( I \) such that (i) \( f^2(J_i) = J_i \), (ii) for each \( i \), there is a point \( x_i \in J_i \) such that \( \{ f^{4n}(x_i) | n \geq 0 \} \) is dense in \( J_i \), and (iii) if \( x \in I - \bigcup_{i=1}^{\infty} J_i \), then \( f^2(x) = x \).

Definitions and terminology. Throughout, \( I \) will denote a closed interval, and \( f : I \to I \) will be a continuous function. If \( n \) is a nonnegative integer, \( f^n \) is the \( n \)-fold composition of \( f \). If \( x \in I \), the orbit of \( x \), \( O(x) \) is \( \{ f^n(x) | n \geq 0 \} \). The statement that \( x \) is periodic means that there is an integer \( n \) such that \( f^n(x) = x \), and the statement that \( x \) has period \( n \) means that \( n \) is a positive integer, \( f^n(x) = x \), and if \( k \) is an integer, \( 1 \leq k < n \), then \( f^k(x) \neq x \).

Associated with \( f : I \to I \) is the inverse limit space

\[
(I, f) = \{(x_0, x_1, \ldots) \mid f(x_{i+1}) = x_i \}
\]

with metric

\[
d((x_0, x_1, \ldots), (y_0, y_1, \ldots)) = \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{2^i}.
\]

In [B-M] and [B-M2] we began a study of the dynamics of functions on the interval by analyzing \( (I, f) \). The theorem in this paper is a continuation of this program.

\( (I, f) \) is a compact, connected metric space, and is an example of what Bing [Bi] has called a snakelike continuum. We will denote elements of \((I, f)\) by subbarred letters, as \( \bar{x} = (x_0, x_1, \ldots) \). The projection maps \( \Pi_n \) of \((I, f)\) onto \( I \) given by \( \Pi_n(\bar{x}) = x_n \) are continuous. If \( H \) is a subcontinuum (compact, connected subspace) of \((I, f)\) we will let \( H_n \) denote \( \Pi_n(H) \). Note that \( H_n \) is a closed interval or point and that \( f(H_{n+1}) = H_n \).

If \( f : I \to I \), then \( f \) induces a homeomorphism \( \hat{f} : (I, f) \to (I, f) \) by \( \hat{f}((x_0, x_1, \ldots)) = (f(x_0), x_0, x_1, \ldots) \). We will utilize the fact that the intersection of any collection of subcontinua of \((I, f)\) is a subcontinuum of \((I, f)\). See [Bi].

If \( S \) is a continuum, the statement that \( S \) is indecomposable means that \( S \) is not the union of two of its proper subcontinua. If \( S \) is indecomposable and \( H \) is a subcontinuum of \( S \), then \( H \) contains no open set in \( S \) [H-Y, pp. 139–141].

Finally, in Lemma 2, we will utilize the following important construction due to Bing [Bi]. Suppose that \( (I, f) \) contains no indecomposable subcontinuum with interior. For each \( \bar{x} \in (I, f) \) let \( g_{\bar{x}} \) be the intersection of all subcontinua of \((I, f)\) that contain interiorly a subcontinuum that contains \( \bar{x} \) in its interior. Then \( g_{\bar{x}} \) is a subcontinuum of \((I, f)\). Furthermore, the sets \( g_{\bar{x}} \) partition \((I, f)\), and if we let \( G = \{ g_{\bar{x}} | \bar{x} \in (I, f) \} \) with the quotient topology, then \( G \) is an arc (i.e. homeomorphic with \( I \)). Moreover, \( f \) induces a homeomorphism \( \hat{f} \) of \( G \) onto \( G \) given by \( \hat{f}(g_{\bar{x}}) = g_{f(\bar{x})} \). Bing also shows that \( g_{\bar{x}} \) does not have interior.

Lemma 1. Suppose that \( g : I \to I \) is continuous, that \((I, g)\) is indecomposable, and that the periodic points of \( g \) are dense in \( I \). Then \( g^2 \) has a dense orbit.
PROOF. We will first argue that \( g \) has a dense orbit. Suppose that \( J \) is a subinterval of \( I \) and that \( x \in \text{int } I \). We will show that there is an integer \( m \) such that \( x \in g^m(J) \). Let \( p \) be a periodic point in \( \text{int } J \). Suppose that \( p \) has period \( k \). Let \( L = \text{cl}(\bigcup_{n=0}^{\infty} g^{-n}(J)) \). Suppose that \( L \neq I \), and that there is a point \( y \in I - L \) such that \( g^k(y) \in L \). Then there is a periodic point \( q \in I - L \) such that \( g^k(q) \in L \). Suppose that the period of \( q \) is \( j \). Then \( g^j(q) = q \), but \( g^j(q) \notin L \). This is a contradiction, and so \( g^{-k}(L) = L \). Then we have \( g^k(L) = L \), and \( g^{-k}(L) = L \). Now, \((I, g)\) is homeomorphic with \((I, g^k)\), but since \( g^{-k}(L) = L \), we have that \((L, g^k)\) is a proper subcontinuum of \((I, g^k)\) with interior. This contradicts the indecomposability of \((I, g)\) \[H-Y, \text{pp. 139–141}\]. Thus,

\[
L = \text{cl}\left(\bigcup_{n=0}^{\infty} g^{-n}(J)\right) = I.
\]

It follows that there is an integer \( m \) such that \( x \in g^m(J) \). This argument shows that if \( x \in \text{int } I \), then \( \bigcup\{g^{-n}(x) | n \geq 0\} \) is dense in \( I \). It follows that if \( U \) is an open set in \( I \), then \( \bigcup_{n=0}^{\infty} g^{-n}(U) | n \geq 0 \) is dense in \( I \). Then using Lemma 3 of \[A-Y\] we have that \( g \) has a dense orbit.

Now, since \( g \) has a dense orbit, and \((I, g)\) is indecomposable, it follows from Theorem 3 of \[B-M\] that \( g^2 \) has a dense orbit.

**Lemma 2.** Suppose that the periodic points of \( f \) are dense, and that \((I, f)\) contains no indecomposable subcontinua with interior. Then \( f^2 \) is the identity.

**Proof.** Since \((I, f)\) contains no indecomposable subcontinua with interior we may use Bing's construction \[Bi\]. That is, there is an upper semicontinuous decomposition space \( G \) of \((I, f)\) into subcontinua and points such that (1) the decomposition space \( G \) is an arc, and (2) no element of \( G \) has interior. Furthermore, the homeomorphism \( \tilde{f} : (I, f) \to (I, f) \) induces a homeomorphism \( \tilde{f} : G \to G \), by \( \tilde{f}(g_x) = g_{\tilde{f}(x)} \).

Since the periodic points of \( f \) are dense, we have that the periodic points of \( \tilde{f} \) are dense, and that the periodic points of \( \tilde{f}^2 \) are dense. Since \( G \) is an arc and \( \tilde{f} \) is a homeomorphism, it follows that \( \tilde{f}^2 \) is the identity.

We will next argue that every element in the decomposition \( G \) is a point. Suppose that \( H \) is a subcontinuum of \((I, f)\) and \( H \in G \). Then \( \tilde{f}^2(H) = H \), and if \( H_0 = \Pi_0(H) \), then \( f^{-2}(H_0) = f^{-2}(\Pi_0(H)) = \Pi_0(f^{-2}(H)) = \Pi_0(H) = H_0 \). Thus, \( H_0 \) is invariant under \( f^2 \). Furthermore, if \( K \in G \), \( K \neq H \), then \( f^2(K_0) = K_0 \), and so \( H_0 \cap K_0 = \emptyset \). Then if \( H \in G \) we have that \( f^{-2}(H_0) = H_0 \). Then \( H = \{(x_0, x_1, \ldots)|x_{2n} \in H_0\} \). Now if \( \text{int } H_0 \neq \emptyset \), then \( H \) has interior. This is a contradiction, and so \( H_0 \) is a point.

Now let \( x \in I \), and let \( \bar{x} \) be a point of \((I, f)\) with \( \Pi_0(\bar{x}) = x_0 = x \). Let \( g_\bar{x} \) be the element of \( G \) containing \( \bar{x} \). Then from the previous discussion we have that \( g_\bar{x} = \{\bar{x}\} \) and that \( \tilde{f}^2(g_\bar{x}) = g_\bar{x} \). Thus, \( \tilde{f}^2(\bar{x}) = \bar{x} \), and so \( f^2(x) = x \). This establishes Lemma 2.
Proof of Theorem. We suppose that the periodic points of \( f \) are dense. Suppose that \( H \) is an indecomposable subcontinuum of \((I, f)\) with interior. Since the periodic points of \( f \) are dense, it follows that the periodic points of \( \hat{f} \) are dense, and so there is a point \( x \in \text{int} \, H \) which is periodic. Suppose that \( x \) has period \( k \).

Consider \( H \cap \hat{f}^k(H) \). This is a subcontinuum of \( H \) with interior, and so \( \hat{f}^k(H) = H \). Then we have that \( f^k(H_0) = f^k(\Pi_0(H)) = \Pi_0(f^k(H)) = \Pi_0(H) = H_0 \).

Consider the intervals \( H_0, H_1, \ldots, H_{k-1} \). We have that \( f(H_i) = H_{i-1} \mod k \). Suppose that \( i \neq j \) and that \( \text{int} \, H_i \cap \text{int} \, H_j \neq \emptyset \). Then we have that \( \hat{f}^j(H) \cap \hat{f}^i(H) \) is a subcontinuum of \( \hat{f}^i(H) \) with interior. Since \( \hat{f}^i(H) \) is indecomposable, we have that \( \hat{f}^j(H) = \hat{f}^i(H) \) and so \( H_i = H_j \). Thus the distinct intervals in \( \{H_0, H_1, \ldots, H_{k-1}\} \) have disjoint interiors.

We will next show that there are at most two distinct intervals in the collection \( \{H_0, H_1, \ldots, H_{k-1}\} \). Suppose the collection contains more than two distinct intervals. A consideration of cases reveals that there are integers \( i, j, r, \) and \( s \) such that \( H_i, H_j, H_r, \) and \( H_s \) are distinct, no interval in the collection lies between \( H_i \) and \( H_j \), and \( H_r \) lies between \( f^j(H_i) \) and \( f^j(H_j) \). Then it follows, since periodic points are dense and \( \text{int} \, H_i \neq \emptyset \), that there is a periodic point \( x \) between \( H_i \) and \( H_j \) such that \( f(x) \in H_r \). That is, \( x \notin \bigcup_{i=0}^{k-1} H_i \), \( f^j(x) \in \bigcup_{i=0}^{k-1} H_i \), and \( f^j(x) = x \) for some \( p > 0 \). But this is impossible since \( f(\bigcup_{i=0}^{k-1} H_i) = \bigcup_{i=0}^{k-1} H_i \). Thus, there are at most two distinct intervals in the collection \( \{H_0, H_1, \ldots, H_{k-1}\} \), and we have \( f^2(H_i) = H_i \) for all \( i \geq 0 \).

Now let \( \{H^1, H^2, H^3, \ldots\} \) be the collection of all indecomposable subcontinua of \((I, f)\) with interior. For each \( i \), let \( J_i = \Pi_0(H^i) \). It follows, as before, that \( \text{int} \, J_i \cap \text{int} \, J_m = \emptyset \) if \( i \neq m \).

From the previous argument we see that \( f^2(J_i) = J_i \). Applying Lemma 1 to \( f^2|J_i: J_i \to J_i \), we see that there is a point \( x_i \in J_i \) such that \( \{f^{4n}(x_i)|n \geq 0\} \) is dense in \( J_i \).

Let \( J = \text{cl}(\bigcup_{n=-1}^\infty J_i) \). Since \( f^2(J_i) = J_i \) for all \( i \geq 1 \), it is clear that if \( x \in J - \bigcup_{i=-1}^\infty J_i \), then \( f^2(x) = x \). Thus the theorem is established in the case that \( J = I \).

Suppose \( J \neq I \). Let \( \{K_{\alpha}\} \) be the collection of all components of \( J \). For each \( \alpha \), \( K_{\alpha} \) is a point or a closed interval. Now let \( L \) be the decomposition space whose elements are the points in \( L - J \) together with the \( K_{\alpha} \). That is, \( L = \{[x]|x \in I\} \) where \( [x] = [y] \) if and only if \( x = y \) or \( x, y \in K_{\alpha} \) for some \( \alpha \). It is clear that \( L \), with the quotient topology, is an arc.

Let \( P: I \to L \) be the projection map, \( P(x) = [x] \). Then \( P \) is a continuous and closed surjection. Furthermore, notice that \( P \) is monotone, i.e. the inverse image of a continuum in \( L \) is a continuum in \( I \). Since \( f \) respects the decomposition there is a uniquely defined continuous surjection \( g: L \to L \) defined by \( g([x]) = [f(x)] \), such that \( g \circ P = P \circ f \). Then \( P \) induces a continuous surjection on inverse limit spaces \( \tilde{P}: (L, f) \to (L, g) \) by \( \tilde{P}((x_0, x_1, \ldots)) = (P(x_0), P(x_1), \ldots) \).

We will now show that \( (L, g) \) contains no indecomposable subcontinuum with interior. Suppose, to the contrary, that \( N \) is an indecomposable subcontinuum of \( (L, g) \) that has nonempty interior. Then, since \( P \) is monotone, it follows that \( \tilde{P} \) is monotone, and hence \( \tilde{P}^{-1}(N) \) is a subcontinuum of \( (I, f) \). It follows from Zorn's Lemma that there is a minimal subcontinuum \( M \) of \( (I, f) \) such that \( P(M) = N \).
If $M$ decomposes as $M = M^1 \cup M^2$, $M^1$ and $M^2$ subcontinua of $M$, then $N = \bar{P}(M^1) \cup \bar{P}(M^2)$. Since $N$ is indecomposable, $\bar{P}(M^1) = N$ or $\bar{P}(M^2) = N$. But then, since $M$ is minimal, $M^1 = M$ or $M^2 = M$. Thus $M$ is indecomposable. We will show that $M$ has interior.

Let $C = \{ x \in (L, g) \mid x_n \neq P(J) \text{ for all } n \geq 0 \}$. Then, since $P(J)$ is closed and has empty interior, $C$ is a closed subset of $(L, g)$ with empty interior. Thus, $\text{int } N \cap ((L, g) - C)$ is a nonempty open subset of $(L, g)$. Each point in $L - P(J)$ has a unique preimage under $P$ and it follows that each point in $(L, g) - C$ has a unique preimage under $\bar{P}$. Thus $\bar{P}^{-1}(\text{int } N \cap ((L, g) - C))$ is an open subset of $M$.

We now have that $M$ is an indecomposable subcontinuum of $(I, f)$ with interior. Thus $M$ must be one of the $H^i$. But then $\bar{P}(M) = N$ is a single point and this contradicts the choice of $N$. Thus, $(L, g)$ has no indecomposable subcontinua with interior.

Since the periodic points under $f$ are dense in $I$, the periodic points under $g$ are dense in $L$. An application of Lemma 2 shows that $g^2$ is the identity on $L$. Thus $f^2$ is the identity on $I - \bigcup_{i=1}^{\infty} J_i$. This completes the proof of the theorem.

**Corollary.** Suppose that the periodic points of $f$ are dense, and that every period of $f$ is a power of 2. Then $f$ is a homeomorphism, and if $x \in I$, then $f^2(x) = x$. In particular, $f$ has no points of period other than 1 or 2.

**Proof.** Suppose that there is a subinterval $J$ of $I$ such that $f^2(J) = J$, and a point $x \in J$ such that $\{ f^{4n}(x) \mid n \geq 0 \}$ is dense in $J$. Then if $g = f^2|J$ we have that $g^2$ has a dense orbit on $J$. It follows from Theorem 13 of [B-M2] that $g$ has a point of odd period. Then, there is a point $x \in J$ which is periodic under $f$, and whose period under $f$ is not a power of 2. This is a contradiction, and so the collection $\{ J_1, J_2, \ldots \}$ in the conclusion of the theorem is empty. It follows then, that if $x \in I$, $f^2(x) = x$. This establishes the corollary.

**References**


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