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DENSE PERIODICITY ON THE INTERVAL

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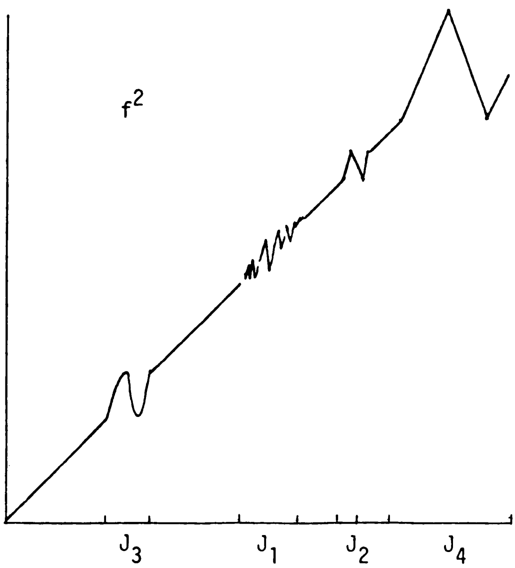
ABSTRACT. We give a description of those continuous functions on the interval for which the set of periodic points is dense.

The purpose of this paper is to describe those continuous functions f on the interval for which the set of periodic points is dense. While this description could have been given in terms of f, it is simpler and more informative to describe f^2.

Suppose that I is an interval, $f: I \to I$ is continuous, and that the set of periodic points of f is dense in I. What follows is a rough description of the graph of f^2: on the diagonal there are "blocks of orbit density," intervals J_i such that $f^2(J_i) = J_i$. For each i, the restriction of f^2 to J_i has a point whose orbit is dense in J_i. Furthermore, if $x \in I - \bigcup_1^\infty J_i$, then $f^2(x) = x$. From this description it follows that if each period of f is a power of 2, then f is a period 2 homeomorphism. See the corollary.



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THEOREM. *Suppose that* $f\colon I \to I$ *is continuous, and that the set of periodic points of* f *is dense in* I. *Then there is a collection (perhaps finite or empty)* $\{J_1, J_2, \dots\}$ *of mutually disjoint closed subintervals of* I *such that* (i) $f^2(J_i) = J_i$, (ii) *for each* i, *there is a point* $x_i \in J_i$ *such that* $\{f^{4n}(x_i) | n \geqslant 0\}$ *is dense in* J_i, *and* (iii) *if* $x \in I - \bigcup_1^\infty J_i$, *then* $f^2(x) = x$.

Definitions and terminology. Throughout, I will denote a closed interval, and $f\colon I \to I$ will be a continuous function. If n is a nonnegative integer, f^n is the n-fold composition of f. If $x \in I$, the *orbit* of x, $O(x)$ is $\{f^n(x) | n \geqslant 0\}$. The statement that x *is periodic* means that there is an integer n such that $f^n(x) = x$, and the statement that x *has period* n means that n is a positive integer, $f^n(x) = x$, and if k is an integer, $1 \leqslant k < n$, then $f^k(x) \neq x$.

Associated with $f\colon I \to I$ is the inverse limit space

$$(I, f) = \{(x_0, x_1, \dots) | f(x_{i+1}) = x_i\}$$

with metric

$$d((x_0, x_1, \dots), (y_0, y_1, \dots)) = \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{2^i}.$$

In **[B-M]** and **[B-M2]** we began a study of the dynamics of functions on the interval by analyzing (I, f). The theorem in this paper is a continuation of this program.

(I, f) is a compact, connected metric space, and is an example of what Bing **[Bi]** has called a snakelike continuum. We will denote elements of (I, f) by subbarred letters, as $\underline{x} = (x_0, x_1, \dots)$. The *projection maps* Π_n, of (I, f) onto I given by $\Pi_n(\underline{x}) = x_n$ are continuous. If H is a *subcontinuum* (compact, connected subspace) of (I, f) we will let H_n denote $\Pi_n(H)$. Note that H_n is a closed interval or point and that $f(H_{n+1}) = H_n$.

If $f\colon I \to I$, then f *induces a homeomorphism* $\hat{f}\colon (I, f) \to (I, f)$ by $\hat{f}((x_0, x_1, \dots)) = (f(x_0), x_0, x_1, \dots)$. We will utilize the fact that the intersection of any collection of subcontinua of (I, f) is a subcontinuum of (I, f). See **[Bi]**.

If S is a continuum, the statement that S is *indecomposable* means that S is not the union of two of its proper subcontinua. If S is indecomposable and H is a subcontinuum of S, then H contains no open set in S **[H-Y**, pp. 139–141].

Finally, in Lemma 2, we will utilize the following important construction due to Bing **[Bi]**. Suppose that (I, f) contains no indecomposable subcontinuum with interior. For each $\underline{x} \in (I, f)$ let $g_{\underline{x}}$ be the intersection of all subcontinua of (I, f) that contain interiorly a subcontinuum that contains $\underline{x}$ in its interior. Then $g_{\underline{x}}$ is a subcontinuum of (I, f). Furthermore, the sets $g_{\underline{x}}$ partition (I, f), and if we let $G = \{g_{\underline{x}} | \underline{x} \in (I, f)\}$ with the quotient topology, then G is an arc (i.e. homeomorphic with I). Moreover, f induces a homeomorphism $\hat{\hat{f}}$ of G onto G given by $\hat{\hat{f}}(g_{\underline{x}}) = g_{\hat{f}(\underline{x})}$. Bing also shows that $g_{\underline{x}}$ does not have interior.

LEMMA 1. *Suppose that* $g\colon I \to I$ *is continuous, that* (I, g) *is indecomposable, and that the periodic points of* g *are dense in* I. *Then* g^2 *has a dense orbit.*

PROOF. We will first argue that g has a dense orbit. Suppose that J is a subinterval of I and that $x \in \operatorname{int} I$. We will show that there is an integer m such that $x \in g^m(J)$. Let p be a periodic point in $\operatorname{int} J$. Suppose that p has period k. Let $L = \operatorname{cl}(\bigcup_{n=0}^{\infty} g^{nk}(J))$. Suppose that $L \neq I$, and that there is a point $y \in I - L$ such that $g^k(y) \in L$. Then there is a periodic point $q \in I - L$ such that $g^k(q) \in L$. Suppose that the period of q is j. Then $g^{jk}(q) = q$, but $g^{jk}(q) \in L$. This is a contradiction, and so $g^{-k}(L) = L$. Then we have $g^k(L) = L$, and $g^{-k}(L) = L$. Now, (I, g) is homeomorphic with (I, g^k), but since $g^{-k}(L) = L$, we have that (L, g^k) is a proper subcontinuum of (I, g^k) with interior. This contradicts the indecomposability of (I, g) **[H-Y**, pp. 139–141]. Thus,

$$L = \operatorname{cl}\left(\bigcup_{n=0}^{\infty} g^{nk}(J)\right) = I.$$

It follows that there is an integer m such that $x \in g^m(J)$. This argument shows that if $x \in \operatorname{int} I$, then $\bigcup\{g^{-n}(x) \mid n \geqslant 0\}$ is dense in I. It follows that if U is an open set in I, then $\bigcup_{n=0}^{\infty}\{g^{-n}(U) \mid n \geqslant 0\}$ is dense in I. Then using Lemma 3 of **[A-Y]** we have that g has a dense orbit.

Now, since g has a dense orbit, and (I, g) is indecomposable, it follows from Theorem 3 of **[B-M1]** that g^2 has a dense orbit.

LEMMA 2. *Suppose that the periodic points of f are dense, and that (I, f) contains no indecomposable subcontinua with interior. Then f^2 is the identity.*

PROOF. Since (I, f) contains no indecomposable subcontinua with interior we may use Bing's construction **[Bi]**. That is, there is an upper semicontinuous decomposition space G of (I, f) into subcontinua and points such that (1) the decomposition space G is an arc, and (2) no element of G has interior. Furthermore, the homeomorphism $\hat{f}\colon (I, f) \to (I, f)$ induces a homeomorphism $\hat{\hat{f}}\colon G \to G$, by $\hat{\hat{f}}(g_{\underline{x}}) = g_{\hat{f}(\underline{x})}$.

Since the periodic points of f are dense, we have that the periodic points of $\hat{f}$ are dense, and that the periodic points of $\hat{\hat{f}}$ are dense. Since G is an arc and $\hat{\hat{f}}$ is a homeomorphism, it follows that $\hat{\hat{f}}^2$ is the identity.

We will next argue that every element in the decomposition G is a point. Suppose that H is a subcontinuum of (I, f) and $H \in G$. Then $\hat{f}^2(H) = H$, and if $H_0 = \Pi_0(H)$, then $f^2(H_0) = f^2(\Pi_0(H)) = \Pi_0(\hat{f}^2(H)) = \Pi_0(H) = H_0$. Thus, H_0 is invariant under f^2. Furthermore, if $K \in G$, $K \neq H$, then $f^2(K_0) = K_0$, and so $H_0 \cap K_0 = \varnothing$. Then if $H \in G$ we have that $f^{-2}(H_0) = H_0$. Then $H = \{(x_0, x_1, \ldots) \mid x_{2n} \in H_0\}$. Now if $\operatorname{int} H_0 \neq \varnothing$, then H has interior. This is a contradiction, and so H_0 is a point.

Now let $x \in I$, and let $\underline{x}$ be a point of (I, f) with $\Pi_0(\underline{x}) = x_0 = x$. Let $g_{\underline{x}}$ be the element of G containing $\underline{x}$. Then from the previous discussion we have that $g_{\underline{x}} = \{\underline{x}\}$ and that $\hat{\hat{f}}^2(g_{\underline{x}}) = g_{\underline{x}}$. Thus, $\hat{f}^2(\underline{x}) = \underline{x}$, and so $f^2(x) = x$. This establishes Lemma 2.

Proof of Theorem. We suppose that the periodic points of f are dense. Suppose that H is an indecomposable subcontinuum of (I, f) with interior. Since the periodic points of f are dense, it follows that the periodic points of $\hat{f}$ are dense, and so there is a point $\underset{\sim}{x} \in \operatorname{int} H$ which is periodic. Suppose that $\underset{\sim}{x}$ has period k.

Consider $H \cap \hat{f}^k(H)$. This is a subcontinuum of H with interior, and so $\hat{f}^k(H) = H$. Then we have that $f^k(H_0) = f^k(\Pi_0(H)) = \Pi_0(\hat{f}^k(H)) = \Pi_0(H) = H_0$.

Consider the intervals $H_0, H_1, \ldots, H_{k-1}$. We have that $f(H_i) = H_{i-1} (\operatorname{mod} k)$. Suppose that $i \neq j$ and that $\operatorname{int} H_i \cap \operatorname{int} H_j \neq \varnothing$. We have then that $\hat{f}^i(H) \cap \hat{f}^j(H)$ is a subcontinuum of $\hat{f}^i(H)$ with interior. Since $\hat{f}^i(H)$ is indecomposable, we have that $\hat{f}^i(H) = \hat{f}^j(H)$ and so $H_i = H_j$. Thus the distinct intervals in $\{H_0, H_1, \ldots, H_{k-1}\}$ have disjoint interiors.

We will next show that there are at most two distinct intervals in the collection $\{H_0, H_1, \ldots, H_{k-1}\}$. Suppose the collection contains more than two distinct intervals. A consideration of cases reveals that there are integers i, j, r, and s such that H_i, H_j, and H_r are distinct, no interval in the collection lies between H_i and H_j, and H_r lies between $f^s(H_i)$ and $f^s(H_j)$. It then follows, since periodic points are dense and $\operatorname{int} H_r \neq \varnothing$., that there is a periodic point x between H_i and H_j such that $f^s(x) \in H_r$. That is, $x \notin \bigcup_{i=0}^{k-1} H_i$, $f^s(x) \in \bigcup_{i=0}^{k-1} H_i$, and $f^p(x) = x$ for some $p > 0$. But this is impossible since $f(\bigcup_{i=0}^{k-1} H_i) = \bigcup_{i=0}^{k-1} H_i$. Thus, there are at most two distinct intervals in the collection $\{H_0, H_1, \ldots, H_{k-1}\}$, and we have $f^2(H_i) = H_i$ for all $i \geqslant 0$.

Now let $\{H^1, H^2, H^3, \ldots\}$ be the collection of all indecomposable subcontinua of (I, f) with interior. For each i, let $J_i = \Pi_0(H^i)$. It follows, as before, that $\operatorname{int} J_i \cap \operatorname{int} J_m = \varnothing$ if $i \neq m$.

From the previous argument we see that $f^2(J_i) = J_i$. Applying Lemma 1 to $f^2|J_i$: $J_i \to J_i$, we see that there is a point $x_i \in J_i$ such that $\{f^{4n}(x_i) | n \geqslant 0\}$ is dense in J_i.

Let $J = \operatorname{cl}(\bigcup_{i=1}^{\infty} J_i)$. Since $f^2(J_i) = J_i$ for all $i \geqslant 1$, it is clear that if $x \in J - \bigcup_{i=1}^{\infty} J_i$, then $f^2(x) = x$. Thus the theorem is established in the case that $J = I$.

Suppose $J \neq I$. Let $\{K_\alpha\}$ be the collection of all components of J. For each α, K_α is a point or a closed interval. Now let L be the decomposition space whose elements are the points in $L - J$ together with the K_α. That is, $L = \{[x] | x \in I\}$ where $[x] = [y]$ if and only if $x = y$ or $x, y \in K_\alpha$ for some α. It is clear that L, with the quotient topology, is an arc.

Let P: $I \to L$ be the projection map, $P(x) = [x]$. Then P is a continuous and closed surjection. Furthermore, notice that P is monotone, i.e. the inverse image of a continuum in L is a continuum in I. Since f respects the decomposition there is a uniquely defined continuous surjection g: $L \to L$ defined by $g([x]) = [f(x)]$, such that $g \circ P = P \circ f$. Then P induces a continuous surjection on inverse limit spaces $\overline{P}$: $(I, f) \to (L, g)$ by $\overline{P}((x_0, x_1, \ldots)) = (P(x_0), P(x_1), \ldots)$.

We will now show that (L, g) contains no indecomposable subcontinua with interior. Suppose, to the contrary, that N is an indecomposable subcontinuum of (L, g) that has nonempty interior. Then, since P is monotone, it follows that $\overline{P}$ is monotone, and hence $\overline{P}^{-1}(N)$ is a subcontinuum of (I, f). It follows from Zorn's Lemma that there is a minimal subcontinuum M of (I, f) such that $P(M) = N$.

If M decomposes as $M = M^1 \cup M^2$, M^1 and M^2 subcontinua of M, then $N = \overline{P}(M^1) \cup \overline{P}(M^2)$. Since N is indecomposable, $\overline{P}(M^1) = N$ or $\overline{P}(M^2) = N$. But then, since M is minimal, $M^1 = M$ or $M^2 = M$. Thus M is indecomposable. We will show that M has interior.

Let $C = \{\underline{x} \in (L, g) | x_n \neq P(J)$ for all $n \geqslant 0\}$. Then, since $P(J)$ is closed and has empty interior, C is a closed subset of (L, g) with empty interior. Thus, int $N \cap ((L, g) - C)$ is a nonempty open subset of (L, g). Each point in $L - P(J)$ has a unique preimage under P and it follows that each point in $(L, g) - C$ has a unique preimage under $\overline{P}$. Thus $\overline{P}^{-1}(\text{int } N \cap ((L, g) - C))$ is an open subset of M.

We now have that M is an indecomposable subcontinuum of (I, f) with interior. Thus M must be one of the H^i. But then $\overline{P}(M) = N$ is a single point and this contradicts the choice of N. Thus, (L, g) has no indecomposable subcontinua with interior.

Since the periodic points under f are dense in I, the periodic points under g are dense in L. An application of Lemma 2 shows that g^2 is the identity on L. Thus f^2 is the identity on $I - \bigcup_{i=1}^{\infty} J_i$. This completes the proof of the theorem.

COROLLARY. *Suppose that the periodic points of f are dense, and that every period of f is a power of* 2. *Then f is a homeomorphism, and if $x \in I$, then $f^2(x) = x$. In particular, f has no points of period other than* 1 *or* 2.

PROOF. Suppose that there is a subinterval J of I such that $f^2(J) = J$, and a point $x \in J$ such that $\{f^{4n}(x) | n \geqslant 0\}$ is dense in J. Then if $g = f^2|J$ we have that g^2 has a dense orbit on J. It follows from Theorem 13 of **[B-M2]** that g has a point of odd period. Then, there is a point $x \in J$ which is periodic under f, and whose period under f is not a power of 2. This is a contradiction, and so the collection $\{J_1, J_2, \ldots\}$ in the conclusion of the theorem is empty. It follows then, that if $x \in I$, $f^2(x) = x$. This establishes the corollary.

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