SHORTER NOTES

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ON THE RADON-NIKODYM PROPERTY
IN BANACH LATTICES

ELIAS SAAB

Abstract. In his recent Memoirs, Talagrand showed that under Axiom L, the weak*-Radon-Nikodym property is equivalent to the Radon-Nikodym property in a Banach lattice. In this note we prove this result by a simple method without assuming Axiom L.

In [3] Talagrand introduced the following notion of the weak*-Radon-Nikodym property:

Definition 1 (Talagrand [3]). A Banach space E has the weak*-Radon-Nikodym property if for every bounded linear operator T: L^1[0,1] → E, there exists a Pettis integrable function ϕ: [0,1] → E** such that

\[ T(f) = \text{Pettis-} \int_0^1 f \phi \, d\lambda \] for every \( f \in L^1[0,1] \).

Let us recall the following

Definition 2. A Banach space E has the Radon-Nikodym property (resp., the weak-Radon-Nikodym property) if for every bounded linear operator T: L^1[0,1] → E there exists a function ϕ: [0,1] → E such that ϕ is Bochner integrable (resp., Pettis integrable) and

\[ T(F) = \text{Bochner-} \int_0^1 f \phi \, d\lambda \] (resp., \( T(f) = \text{Pettis-} \int_0^1 f \phi \, d\lambda \))

for every \( f \in L^1[0,1] \).

In [3, pp. 88–91] Talagrand showed that under Axiom L a Banach lattice E has the Radon-Nikodym property if and only if it has the weak*-Radon-Nikodym property. His proof, besides the fact that it uses Axiom L, is quite involved. (Axiom L states that [0,1] cannot be covered by less than the continuum of closed negligible sets.)

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The following more general theorem is true and its proof is quite simple.

**Theorem 3.** For any Banach lattice $E$, the following statements are equivalent:

1. The space $E$ has the Radon-Nikodym property.
2. The space $E$ has the weak-Radon-Nikodym property.
3. The space $E$ has the weak*-Radon-Nikodym property.

**Proof.** (1) is equivalent to (2) by Ghoussoub and Saab [1]. So it is enough to show that (3) $\implies$ (2). (3) implies that every vector measure of bounded variation with values in $E$ has relatively compact range by a result of Stegall, and therefore $c_0$ does not embed in $E$ (see [1]). Hence $E$ is weakly sequentially complete and therefore $E$ is complemented in its bidual $E^{**}$ by a projection

$$P: E^{**} \to E \quad \text{(see [2, p. 34]).}$$

Let $T: L^1[0,1] \to E$ be a bounded linear operator. By (3) there is a function $\phi: [0,1] \to E^{**}$, Pettis integrable such that

$$T(f) = \text{Pettis-} \int_0^1 f\phi \, d\lambda$$

for every $f$ in $L^1[0,1]$. The function $P\phi: [0,1] \to E$ is Pettis integrable and

$$T(F) = P(T(f)) = \text{Pettis-} \int_0^1 f P\phi \, d\lambda.$$

So $T$ has a Pettis kernel in $E$ and therefore $E$ has the weak-Radon-Nikodym property.

**References**


Department of Mathematics, University of Missouri-Columbia, Columbia, Missouri 65211