RELATIVE LUBIN-TATE GROUPS
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ABSTRACT. We construct a class of formal groups that generalizes Lubin-
Tate groups. We formulate the major properties of these groups and indicate
their relation to local class field theory.

The aim of this note is to introduce a certain family of formal groups generalizing
Lubin-Tate groups. Although the construction, basic properties and relation with
local class field theory are all similar to Lubin-Tate theory, the author is unaware
of previous references to these groups. We remark, however, that they are comple-
mentary in some sense to the formal groups studied by Honda in [2]. Since we want
to keep this note short, all the proofs are omitted. The reader who is acquainted
with Lubin-Tate theory as in [4 or 5] will be able to supply them without any
difficulties.

I would like to acknowledge my debt to K. Iwasawa. His beautiful exposition of
local class field theory [3] motivated this note.

1. Let \( k \) be a finite extension of \( \mathbb{Q}_p \), \( \nu: k^\times \to \mathbb{Z} \) the normalized valuation
(normalized in the sense that \( \nu(k^\times) = \mathbb{Z} \)), \( \mathcal{O} \) and \( \mathfrak{p} \) its ring of integers and maximal
ideal, and \( \overline{k} = \mathcal{O}/\mathfrak{p} \) the residue field, a finite field of characteristics \( p \) and \( q \) elements.
\( k_{\text{algs}} \) denotes an algebraic closure of \( k \) and \( k_{\text{ur}} \) the maximal unramified extension of
\( k \) in it. We also fix a completion of \( k_{\text{algs}} \), \( \Omega \), and let \( K \) be the closure of \( k_{\text{ur}} \) in it.
We write \( \varphi \) for the Frobenius automorphism of \( k_{\text{ur}}/k \), characterized by \( \varphi(x) = x^q \mod \mathfrak{p} \),
for all \( x \in \mathcal{O}_{k_{\text{ur}}} \). It extends by continuity to an automorphism of \( K/k \),
still denoted by \( \varphi \). If \( k' \) is another finite extension of \( \mathbb{Q}_p \), the corresponding objects
will be denoted by \( ' \), e.g. \( \varphi', q' \), etc.

If \( A \) is any ring, \( A[[X_1, \ldots, X_n]] \) will denote the power series ring in \( X_i \). If \( f \) and
\( g \) are elements of it, \( f \equiv g \mod \deg m \) means that the power series \( f - g \) involves
only monomials of degree at least \( m \).

2. Fix the field \( k \). For each integer \( d \) let \( \Sigma_d \) be the set of all \( \xi \in k \), \( \nu(\xi) = d \).
Fix also \( d > 0 \) and let \( k' \) be the unique unramified extension of \( k \) of degree \( d \). Let
\( \xi \in \Sigma_d \) and consider

\[\mathcal{F}_\xi = \{ f \in \mathcal{O}'[[X]] \mid f \equiv \pi' X \mod \deg 2, \ N_{k'/k}(\pi') = \xi \text{ and } f \equiv X^q \mod \mathfrak{p}' \} .\]

**Theorem 1.** For each \( f \in \mathcal{F}_\xi \) there is a unique one-dimensional commutative
formal group law \( F_f \in \mathcal{O}'[[X, Y]] \) satisfying \( F_f^\varphi \circ f = f \circ F_f \). In others words, \( f \) is
a homomorphism of \( F_f \) to \( F_f^\varphi \).

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Note that if \( f \in \mathcal{F}_\xi \), \( f^\varphi \in \mathcal{F}_\xi \) also, and necessarily \( F_f^\varphi = F_{\varphi(f)} \). If \( d = 1 \), we are in the situation considered by Lubin and Tate. In general, we call \( F_f \) a relative Lubin-Tate group (relative to the extension \( k'/k \)).

3.

**THEOREM 2.** Let \( f = \pi'X + \cdots, g = \pi''X + \cdots \) be in \( \mathcal{F}_\xi \). Let \( a \in \mathcal{O}' \) be an element for which \( \alpha^\varphi = \alpha''/\alpha' \). Then there exists a unique power series \( [a]_{f,g} \in \mathcal{O}'[[X]] \) for which

(i) \( [a]_{f,g} = aX \mod \deg 2 \),

(ii) \( [a]_{f,g} \circ f = g \circ [a]_{f,g} \).

\([a]_{f,g} \) is therefore in \( \text{Hom}(F_f, F_g) \). If \( h = \pi'''X + \cdots \) and \( b^\varphi = \pi''/\pi''' \), \( [b]_{f,h} = [b]_{g,h} \circ [a]_{f,g} \). Moreover, the map \( \alpha \mapsto [a]_{f,g} \) is an additive injective homomorphism from \( \{ \alpha \in \mathcal{O}' | \alpha^\varphi = \alpha''/\alpha' \} \) to \( \text{Hom}(F_f, F_g) \). If \( f = g \) it is a ring homomorphism \( \mathcal{O} \to \text{End}(F_f) \), \( \alpha \mapsto [a]_f = [a]_{f,f} \).

**COROLLARY.** If \( f, g \in \mathcal{F}_\xi \), \( F_f \) and \( F_g \) are isomorphic.

4. Pick \( \xi, \xi' \in \Sigma_d \) and set \( v = \xi/\xi' \). Let \( u \) be a unit of \( k' \) such that \( N_{k'/k}(u) = v \), \( \theta_1 \in K \) such that \( \theta_1 \theta_1^{-1} = \xi/\xi' \), and \( f \in \mathcal{F}_\xi \).

**THEOREM 3.** There exists a unique power series \( \theta(X) \in \mathcal{O}_K[[X]] \) satisfying

(i) \( \varphi(\theta) = \theta \circ [v]_f \),

(ii) \( \theta(X) \equiv \theta_1 X \mod \deg 2 \).

Put \( f' = \varphi^\varphi \circ f \circ \theta^{-1} \). Then \( f' \in \mathcal{F}_\xi \), and \( \theta \) is an isomorphism of \( F_f \) onto \( F_{f'} \) over \( \mathcal{O}_K \).

5.

**DEFINITION.** For \( i \geq 0 \) and \( f \in \mathcal{F}_\xi \), let \( f^{(i)} = \varphi^{i-1}(f) \circ \cdots \circ \varphi(f) \circ f \). Then \( f^{(i)} \in \text{Hom}(F_f, F_f^\varphi) \) and (if \( \xi \in \Sigma_d \)) \( f^{(d)} = [\xi]_f \in \text{End}(F_f) \). Note also that \( \varphi^i(f^{(i)}) \circ f^{(j)} = f^{(i+j)} \).

Let \( M \) be the valuation ideal of \( \Omega \), and \( M_f \) the commutative group whose underlying set is \( \mathcal{M} \) and the addition is given by \( F_f \). With \( \xi \in \Sigma_d, f \in \mathcal{F}_\xi \) and \( \pi \) a prime element of \( \mathcal{O} \), define for any \( n \geq 0 \)

\[
W_f^n = \{ \alpha \in M_f | [a]_f(\alpha) = 0 \text{ for all } a \in \varphi^{n+1} \}
= \{ \alpha \in M_f | [\varphi^{n+1}]_f(\alpha) = 0 \}
= \text{Ker}(f^{(n+1)}: M_f \to M^{\varphi^{n+1}}(f)).
\]

**PROPOSITION 1.** (i) \( W_f^n \) is a finite sub-\( \mathcal{O} \)-module of \( M_f \) and has \( q^{n+1} \) elements. \( W_f^n \subseteq W_f^{n+1} \).

(ii) If \( \alpha \in W_f^n \) but \( \alpha \not\in W_f^{n-1} \), then \( \alpha \mapsto [a]_f(\alpha) \) gives an isomorphism \( \mathcal{O}/\varphi^{n+1} \cong W_f^n \).

(iii) \( W_f = \bigcup W_f^n \cong k/\mathcal{O} \) (noncanonically) and is the set of all \( \mathcal{O} \)-torsion in \( M_f \).

6. Coleman’s norm operator (see [1]). Let \( R = \mathcal{O}'[[X]], \xi \in \Sigma_d \), and \( f \in \mathcal{F}_\xi \).

**PROPOSITION 2.** There exists a unique multiplicative operator \( \mathcal{N}: R \to R \) (\( \mathcal{N} = \mathcal{N}_f \), to emphasize the dependence on \( f \)), such that

\[
(\mathcal{N} h) \circ f(X) = \prod_{a \in W_f^0} h(X[+_f a]) \forall h \in R.
\]
It enjoys the additional properties:

(i) $\mathcal{N}h \equiv h^p \mod p'$,
(ii) $\mathcal{N}_f \varphi = \varphi \circ \mathcal{N}_f \circ \varphi^{-1}$, i.e. $\mathcal{N}_f \varphi(h^p) = (\mathcal{N}_f h)^p$,
(iii) Let $\mathcal{N}^{(i)}_f h = \mathcal{N}_{\varphi^{-1}(f)} \circ \cdots \circ \mathcal{N}_{\varphi(f)} \circ \mathcal{N}_f(h)$.

Then

$$(\mathcal{N}^{(i)}_f h) \circ f^{(i)}(X) = \prod_{\alpha \in W^{i-1}_f} h(X^{\alpha})$$

(iv) If $h \in R$ and $h \equiv 1 \mod \varphi^i$ ($i \geq 1$), then $\mathcal{N}h \equiv 1 \mod \varphi^{i+1}$.

7.

**Proposition 3.** The field $k'(W^f_j)$ is the same for all $f \in \mathcal{F}_\xi$. Call it $k'^{\xi}$, and put $k^{\xi-1}_\xi = k'$. Then for $n \geq 0$, $k^{\xi n}_\xi$ is a totally ramified extension of $k'$ of degree $(q-1)q^n$, and it is abelian over $k$. Any $\alpha$ in $W^{n}_f$ but not in $W^{n-1}_f$, for any $f \in \mathcal{F}_\xi$, generates $k^{\xi n}_\xi$ over $k'$ and is a prime element for it.

Much more can be said about those fields (see §10).

8. Coleman power series [1].

**Theorem 4.** Fix $\xi \in \Sigma_d$, $f \in \mathcal{F}_\xi$ and $\alpha \in W^{n-1}_f$, $\alpha \not\in W^{n-1}_{\varphi^{-1}(f)}$. For $0 \leq i \leq n$ let $\alpha_i = (\varphi^{-i}(f))^{(n-i)}(\alpha) = \varphi^{-i-1}(f) \circ \cdots \circ \varphi^{-i}(f)(\alpha) \in W^{i-1}_{\varphi^{-i}(f)}$. Let $c$ be a unit of $k^{\xi n}_\xi$ and $c_i = N_{n,i}(c)$ ($N_{n,i}$ denoting the norm from $k^{\xi n}_\xi$ to $k^{\xi i}_\xi$). Then there is a power series $g$ in $R$ such that

$$\varphi^{-i}(g)(\alpha_i) = c_i \quad (0 \leq i \leq n).$$

**Corollary.** Suppose $\alpha_i$ is an element of $W^{i-1}_{\varphi^{-i}(f)}$ not in $W^{i-1}_{\varphi^{-i-1}(f)}$ ($i \geq 0$) and $f^{\varphi^{-i}}(\alpha_{i+1}) = \alpha_{i+1}$. Suppose also $c_0, c_1, \ldots$ is a norm-compatible sequence of units in $k^{\xi n}_\xi$, i.e. $N_{n,i}(c_n) = c_i$. Then there exists a unique $g$ in $R$ such that $g^{\varphi^{-i}}(\alpha_i) = c_i$ for all $i$.

9.

**Example.** Let $K$ be a quadratic imaginary field, let $F$ be a finite extension of $K$, and let $E$ be an elliptic curve defined over $F$ with complex multiplication by the full ring of integers of $K$. As explained in [6], if we choose a Weierstrass model of $E$ over the integers of $F$ we get a formal group law $\hat{E}(X, Y)$ defined over the ring generated (over $\mathbb{Z}$) by the coefficients in the Weierstrass equation. Let $p$ be a prime of $K$ and $P$ a prime of $F$ dividing $p$. Assume $E$ has good reduction at $P$, and that $P$ is not ramified in $F/K$. It is then a consequence of the theory of complex multiplication that $\hat{E}$, as a formal group defined over $\mathcal{O}_P$ (the integers of $F_P$), is a relative Lubin-Tate group with respect to the (unramified) extension $F_P/K_p$.

10. The relation between Lubin-Tate groups and local class field theory can now be easily generalized. A full description of it (and actually derivation of local class field theory from the formal group point of view) can be found in [3]. We only make the following remarks. The fields $k_\xi = \bigcup k^{\xi n}_\xi = k'(W^f_j)$ (for any $f \in \mathcal{F}_\xi$) are the maximal abelian extensions of $k$ with residue field equal to the extension of degree $d$ of $\bar{k}$. They are distinct for different $\xi$ as can be seen from the observation that the group of universal norms from $k_\xi$ to $k$ is just the cyclic group generated by $\xi$. 

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If $\xi \in \Sigma_1^d$, i.e. is a $d$th power in $k$, then $\mathcal{F}_\xi$ contains an $f$ from $\mathcal{O}[[X]]$. In this case $k\xi$ is the compositum of a totally ramified extension of $k$ and $k'$. However, this is not always the case, because $\Sigma_d \neq \Sigma_1^d$ in general.

REFERENCES


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