A FOOTNOTE TO THE MULTIPLICATIVE BASIS THEOREM

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ABSTRACT. We characterize those perfect fields $k$ such that for each integer $n \geq 1$, but there are but finitely many isomorphism types of $k$-algebras of dimension $n$ that are of finite representation type. Some remarks on the imperfect case are also presented.

A finite-dimensional algebra $A$ over a field $k$ is of finite representation type if it has only finitely many isomorphism types of indecomposable modules. A multiplicative basis for $A$ is a $k$-basis $B$ such that $B \cup \{0\}$ is a semigroup under the multiplication in $A$. Recently, Bautista, Gabriel, Roiter and Salmerón [1] have shown that an algebraically closed field $k$ has the property that each $k$-algebra of finite representation type has a multiplicative basis. Let us point out at once that this property characterizes algebraically closed fields. For, if a field $k$ has a finite extension field $F$, then $F$ is of finite representation type, and any multiplicative basis $B$ for $F$ would be a finite semigroup with cancellation, hence a group. Then, $F$ would be isomorphic to the group algebra $kB$, but $kB$ can never be simple if $B$ is nontrivial.

An important consequence of the theorem above is that when $k$ is algebraically closed, there are but finitely many $k$-algebras of finite representation type of any fixed $k$-dimension ("finite representation type is finite"). Let us express this by saying that $k$ has property (N). We wish here to discuss other fields with property (N).

Recall from [3, III-29] that a field $k$ is of type (F) if it is perfect and, for each $n \geq 1$, there are only finitely many $k$-isomorphism types of field extensions of degree $n$ over $k$. Examples are finite fields, local fields with finite residue field and the fields $F((T))$ of quotients of power series over algebraically closed fields $F$ of characteristic zero.

THEOREM. A perfect field $k$ has property (N) if and only if it is of type (F).

PROOF. One implication is clear. If $k$ is of type (F), let $\bar{k}$ be an algebraic closure of $k$, and let $G$ be the Galois group of $\bar{k}/k$. By [2], a finite-dimensional $k$-algebra $A$ is of finite representation type if and only if $\bar{k} \otimes_k A$ is. Hence, it suffices to show that a $\bar{k}$-algebra $\mathcal{A}$ of the form $\bar{k} \otimes_k A$ has only finitely many $k$-forms, up to isomorphism. Such forms are classified by the set $H^1(G, \text{Aut}_{k\text{-alg}}(\mathcal{A}))$. This set is finite by [3, III-30] since $\text{Aut}_{k\text{-alg}}(\mathcal{A})$ is a linear algebraic group defined over $k$, and the proof is complete.

Let us remark on the case of imperfect fields. The degree of imperfection of a field $k$ of characteristic $p$ is that integer $r$ so that $[k : k^p] = p^r$ (or infinity, if $[k : k^p]$ is infinite). Thus, $k$ is perfect just when its degree of imperfection is zero. It is well
known that if \( k \) has degree of imperfection two or more, then \( k \) has infinitely many nonisomorphic extensions of degree \( p \) (see [4, II.11.6]). Thus, an imperfect field with property (N) must have degree of imperfection one. The best known examples of such fields are the function fields in one variable over a field of characteristic \( p \). These, however, usually have many separable extensions. The most likely candidate for an imperfect field with property (N) would seem to be the separable closure of \( F((T)) \), with \( F \) algebraically closed of finite characteristic. It should be noted that inseparable base changes generally destroy finite representation type, so descent methods are not likely to resolve the question.

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REFERENCES