SOME ALGEBRAIC SETS OF HIGH LOCAL COHOMOLOGICAL DIMENSION IN PROJECTIVE SPACE

GENNADY LYUBEZNK

Abstract. Let $V_0, \ldots, V_{[n/t]}$ be algebraic sets of pure codimension $t$ in $P^n$, and suppose $\bigcap V_i$ is empty. Then $P^n - \bigcup V_i$ has cohomological dimension $n - [n/t]$.

If $U$ is a scheme, then $\text{cd}(U)$, the cohomological dimension of $U$, is the largest integer $i$ such that there exists a quasi-coherent sheaf $F$ on $U$ such that $H^i(F) \neq 0$.

In [1], G. Faltings proved that if $V$ is an algebraic set of pure codimension $t$ in $P^n$, then

$$\text{cd}(P^n - V) \leq n - [n/t].$$

This note gives some algebraic sets for which equality holds in (1).

Theorem. Put $s = [n/t]$ and let $V = V_0 \cup V_1 \cup \cdots \cup V_s$ be the union of $s + 1$ algebraic sets of pure codimension $t$ in general position in $P^n$ (i.e. such that the intersection of all of them is empty). Then

$$\text{cd}(P^n - V) = n - [n/t].$$

This theorem (from the author’s thesis [4]) answers the conjecture from [3] in the affirmative and covers all three examples from [3], but not the statement of the main theorem.

For a proof it is convenient to translate the problem into an algebraic language. Put $R_n = k[x_0, \ldots, x_n]$ and let $\mathfrak{A}$ be the defining ideal of $V$ in $R_n$. Then the cohomological dimension of $P^n - V$ is the largest integer $i$ such that $H^i(R_n) \neq 0$ (cf. [2]).

Lemma. Put $s = [n/t]$ and let $\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_j$ be $j + 1$ homogeneous ideals of pure height $t$ in $R_n$. Put $\beta_j = \Sigma_{r=0}^{\infty} \mathfrak{A}_r$. Then $H^i_{\beta_r}(R_n) = 0$ if $i \geq n - s + j + 2$.

Proof. If $j = 0$, the result follows from (1). Put $\beta_{j-1} = \Sigma_{r=0}^{\infty} \mathfrak{A}_r$. Then $\beta_{j-1} \cap \mathfrak{A}_j$ has the same radical as $\gamma_{j-1} = \Sigma_{r=0}^{\infty} (\mathfrak{A}_r \cap \mathfrak{A}_j)$. Since $\beta_{j-1}$ and $\gamma_{j-1}$ are sums of $j - 1$ ideals of pure heights $t$ in $R_n$, we may assume that $H^i_{\beta_{j-1}}(R_n) = H^i_{\gamma_{j-1}}(R_n) = 0$ for all $i \geq n - s + j + 1$. We also know that $H^i_{\beta_j}(R_n) = 0$ if $i \geq n - s + 2$. The Mayer-Vietoris long exact sequence gives

$$H^i_{\gamma_{j-1}}(R_n) \rightarrow H^i_{\beta_{j-1}}(R_n) \rightarrow H^i_{\beta_j}(R_n) \oplus H^i_{\beta_j}(R_n)$$

and this proves the Lemma.

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Proof of the Theorem. Let \( \mathfrak{A}_0, \ldots, \mathfrak{A}_s \) be the defining ideals of \( V_0, \ldots, V_s \) in \( R_n \). Put \( \mathfrak{E}_j = \mathfrak{A}_0 \cap \mathfrak{A}_1 \cap \cdots \cap \mathfrak{A}_j + \mathfrak{A}_{j+1} + \cdots + \mathfrak{A}_s \). Then the biggest integer \( i \) for which \( H^n_{\mathfrak{E}_j}(R_n) \neq 0 \) is \( i = n - j + 1 \). We are going to prove this by induction on \( j \) and the theorem will follow by putting \( j = s \).

For \( j = 0 \), \( \mathfrak{E}_j \) is \( m \)-primary, where \( m \) is the maximal ideal of \( R_n \) and the above claim is well known in this case. Assume \( j > 0 \) and assume the Theorem proven for \( j - 1 \). Put \( \mathfrak{A}' = \mathfrak{A}_j + \mathfrak{A}_{j+1} + \cdots + \mathfrak{A}_s \) and \( \mathfrak{A}'' = \mathfrak{A}_0 \cap \mathfrak{A}_1 \cap \cdots \cap \mathfrak{A}_{j-1} + \mathfrak{A}_{j+1} + \mathfrak{A}_{j+2} + \cdots + \mathfrak{A}_s \). Then \( \mathfrak{E}_j = \mathfrak{A}' \cap \mathfrak{A}'' \) and \( \mathfrak{E}_{j-1} = \mathfrak{A}' + \mathfrak{A}'' \). By the Lemma \( H^n_{\mathfrak{E}_j}(R_n) = H^n_{\mathfrak{A}''}(R_n) = 0 \) for all \( i \geq n - j + 2 \). The claim now follows from the Mayer-Vietoris sequence considering the induction hypothesis. Q.E.D.

Remark. The above Lemma gives a lower bound on the number of algebraic sets of given codimension which are needed to cut out a given algebraic subset of \( \mathbb{P}^n \) set-theoretically. Namely, if \( V \subset \mathbb{P}^n \) and \( \text{cd}(\mathbb{P}^n - V) = v \), then we need at least \( v + 1 - (n - \lfloor n/t \rfloor) \) algebraic subsets of pure codimension \( t \) to cut out \( V \) set-theoretically.

Faltings' inequality (1) and the fact that it is exact for all \( n \) and \( t \) (our Theorem) suggest the following.

Conjecture. Every algebraic subset of \( \mathbb{P}^n \) of pure codimension \( t \) is a set-theoretic intersection of \( n + 1 - \lfloor n/t \rfloor \) hypersurfaces \([4, \text{p. 8}]\).

For additional supporting evidence for this conjecture see \([4, \text{Theorem 6}]\).

References

Department of Mathematics, Columbia University, New York, New York 10027

Current address: Department of Mathematics, Purdue University, West Lafayette, Indiana 47907