SOME ALGEBRAIC SETS OF HIGH LOCAL COHOMOLOGICAL
DIMENSION IN PROJECTIVE SPACE

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Abstract. Let \( V_0, \ldots, V_{\lfloor n/t \rfloor} \) be algebraic sets of pure codimension \( t \) in \( P^n \), and suppose \( \bigcap V_i \) is empty. Then \( P^n - \bigcup V_i \) has cohomological dimension \( n - \lfloor n/t \rfloor \).

If \( U \) is a scheme, then \( \text{cd}(U) \), the cohomological dimension of \( U \), is the largest integer \( i \) such that there exists a quasi-coherent sheaf \( F \) on \( U \) such that \( H^i(F) \neq 0 \).

In [1], G. Faltings proved that if \( V \) is an algebraic set of pure codimension \( t \) in \( P^n \), then

\[
\text{cd}(P^n - V) \leq n - \lfloor n/t \rfloor.
\]

This note gives some algebraic sets for which equality holds in (1).

Theorem. Put \( s = \lfloor n/t \rfloor \) and let \( V = V_0 \cup V_1 \cup \cdots \cup V_s \) be the union of \( s + 1 \) algebraic sets of pure codimension \( t \) in general position in \( P^n \) (i.e. such that the intersection of all of them is empty). Then

\[
\text{cd}(P^n - V) = n - \lfloor n/t \rfloor.
\]

This theorem (from the author’s thesis [4]) answers the conjecture from [3] in the affirmative and covers all three examples from [3], but not the statement of the main theorem.

For a proof it is convenient to translate the problem into an algebraic language. Put \( R_n = k[x_0, \ldots, x_n] \) and let \( \mathfrak{A} \) be the defining ideal of \( V \) in \( R_n \). Then the cohomological dimension of \( P^n - V \) is the largest integer \( i \) such that \( H^{i+1}(\mathfrak{A}) \neq 0 \) (cf. [2]).

Lemma. Put \( s = \lfloor n/t \rfloor \) and let \( \mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_j \) be \( j + 1 \) homogeneous ideals of pure height \( t \) in \( R_n \). Put \( \beta_j = \sum_{i=0}^{s} \mathfrak{A}_i \). Then \( H_i^{\beta_j}(R_n) = 0 \) if \( i \geq n - s + j + 2 \).

Proof. If \( j = 0 \), the result follows from (1). Put \( \beta_{j-1} = \sum_{i=0}^{s-1} \mathfrak{A}_i \). Then \( \beta_{j-1} \cap \mathfrak{A}_j \) has the same radical as \( \gamma_{j-1} = \sum_{i=0}^{s-1} (\mathfrak{A}_i \cap \mathfrak{A}_j) \). Since \( \beta_{j-1} \) and \( \gamma_{j-1} \) are sums of \( j - 1 \) ideals of pure heights \( i \) in \( R_n \), we may assume that \( H_i^{\beta_{j-1}}(R_n) = H_i^{\gamma_{j-1}}(R_n) = 0 \) for all \( i \geq n - s + j + 1 \). We also know that \( H_i^{\mathfrak{A}_j}(R_n) = 0 \) if \( i \geq n - s + 2 \). The Mayer-Vietoris long exact sequence gives

\[
H_{\gamma_{j-1}}^{i+1}(R_n) \to H_{\beta_{j-1}}^{i+1}(R_n) \to H_{\beta_{j-1}}^{i+1}(R_n) \oplus H_{\mathfrak{A}_j}^{i+1}(R_n)
\]

and this proves the Lemma.
Proof of the Theorem. Let \( \mathfrak{A}_0, \ldots, \mathfrak{A}_s \) be the defining ideals of \( V_0, \ldots, V_s \) in \( R_n \). Put \( \mathfrak{S}_j = \mathfrak{A}_0 \cap \cdots \cap \mathfrak{A}_j + \mathfrak{A}_{j+1} + \cdots + \mathfrak{A}_s \). Then the biggest integer \( i \) for which \( H^i_{\mathfrak{S}_j}(R_n) \neq 0 \) is \( i = n - j + 1 \). We are going to prove this by induction on \( j \) and the theorem will follow by putting \( j = s \).

For \( j = 0 \), \( \mathfrak{S}_j \) is \( m \)-primary, where \( m \) is the maximal ideal of \( R_n \) and the above claim is well known in this case. Assume \( j > 0 \) and assume the Theorem proven for \( j - 1 \). Put \( \mathfrak{A}' = \mathfrak{A}_j + \mathfrak{A}_{j+1} + \cdots + \mathfrak{A}_s \) and \( \mathfrak{A}'' = \mathfrak{A}_0 \cap \mathfrak{A}_1 \cap \cdots \cap \mathfrak{A}_{j-1} + \mathfrak{A}_{j+1} + \mathfrak{A}_{j+2} + \cdots + \mathfrak{A}_s \). Then \( \mathfrak{S}_j = \mathfrak{A}' \cap \mathfrak{A}'' \) and \( \mathfrak{S}_{j-1} = \mathfrak{A}' + \mathfrak{A}'' \). By the Lemma \( H^i_{\mathfrak{S}_j}(R_n) = H^i_{\mathfrak{S}_{j-1}}(R_n) = 0 \) for all \( i \geq n - j + 2 \). The claim now follows from the Mayer-Vietoris sequence considering the induction hypothesis. Q.E.D.

Remark. The above Lemma gives a lower bound on the number of algebraic sets of given codimension which are needed to cut out a given algebraic subset of \( \mathbb{P}^n \) set-theoretically. Namely, if \( V \subset \mathbb{P}^n \) and \( \text{cd}(\mathbb{P}^n - V) = v \), then we need at least \( v + 1 - (n - \lfloor n/t \rfloor) \) algebraic subsets of pure codimension \( t \) to cut out \( V \) set-theoretically.

Faltings’ inequality (1) and the fact that it is exact for all \( n \) and \( t \) (our Theorem) suggest the following.

Conjecture. Every algebraic subset of \( \mathbb{P}^n \) of pure codimension \( t \) is a set-theoretic intersection of \( n + 1 - \lfloor n/t \rfloor \) hypersurfaces [4, p. 8].

For additional supporting evidence for this conjecture see [4, Theorem 6].

References


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