PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 95, Number 1, September 1985

ON ALMOST EVERYWHERE CONVERGENCE OF BOCHNER-RIESZ MEANS IN HIGHER DIMENSIONS

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ABSTRACT. In $\mathbf{R}^n$ define $(T_{\lambda,r}f)\widehat{\ }(\xi) = \hat{f}(\xi)(1 - |r^{-1}\xi^2|)_+^\lambda$. If $n \geq 3$, $\lambda > \frac{1}{2}(n-1)/(n+1)$ and $2 \leq p < 2n/(n-1-2\lambda)$, then $\lim_{r\to\infty} T_{\lambda,r}f(x) = f(x)$ a.e. for all $f \in L^p(\mathbf{R}^n)$.

The Bochner-Riesz operators in $\mathbf{R}^n$ are defined as $(T_\lambda f)\widehat{\ }(\xi) = \hat{f}(\xi)(1 - |\xi|^2)_+^\lambda$, and the associated maximal operators are

$$T_\lambda^* f(x) = \sup_{r>0} |(\hat{f} \cdot (1 - |r\xi|^2)_+^\lambda)^\vee|(x).$$

It is conjectured that, when $\lambda > 0$, T_λ is bounded on L^p if and only if $p_0'(\lambda) < p < p_0(\lambda)$, where $p_0(\lambda) = 2n/(n-1-2\lambda)$. That the restrictions on p and λ are necessary was shown by Herz [**7**]. Carleson and Sjölin [**3**] proved the conjecture in dimension two. Moreover, in $\mathbf{R}^2$ Carbery [**1**] has established boundedness of T_λ^* on L^p, and hence almost everywhere convergence of Bochner-Riesz means, for the same range of p and λ except for the added restriction $p \geq 2$. In dimensions greater than two it is known by work of C. Fefferman, Stein and Tomas [**5, 6, 10**] that T_λ is bounded on L^p, provided $p_0'(\lambda) < p < p_0(\lambda)$ and $\lambda > \frac{1}{2}(n-1)/(n+1)$, but the remaining cases have not been resolved. Our principal result is

THEOREM 1. *T_λ^* is bounded on $L^p(\mathbf{R}^n)$ whenever $2 \leq p < p_0(\lambda)$ and $\lambda > \frac{1}{2}(n-1)/(n+1)$ for all $n \geq 3$.*

Interest in L^p bounds for T_λ^* is due to the consequence

COROLLARY. *If $f \in L^p(\mathbf{R}^n)$, $n \geq 3$, $\lambda > \frac{1}{2}(n-1)/(n+1)$ and $2 \leq p < p_0(\lambda)$, then*

$$\lim_{r\to\infty} (\hat{f} \cdot (1 - |\xi/r|^2)_+^\lambda)^\vee(x) = f(x) \quad a.e.$$

The proof is based on the L^2 restriction theorem of Tomas and Stein [**10**]. Our second result is a small extension of that theorem, related in spirit to Theorem 1.

THEOREM 2. *Suppose μ is a nonnegative radial measure on $\mathbf{R}^n$, satisfying $\mu(\{0\}) = 0$, and $n \geq 2$. Let $\gamma = n(n-1)/(n+1)$. Suppose $1 < p \leq 2(n+1)/(n+3)$ and $q = ((n-1)/(n+1))p'$. Then a necessary and sufficient condition that the weighted norm inequality $\|\hat{f}\|_{L^q(\mathbf{R}^n, d\mu)} \leq C\|f\|_{L^p(\mathbf{R}^n)}$ hold is that there exist $A < \infty$ such that, for each $0 < r < \infty$, $\mu\{r < |\xi| < 2r\} \leq Ar^\gamma$.*

Received by the editors August 9, 1984.
1980 *Mathematics Subject Classification*. Primary 42B15, 42B25.

To begin the proof of Theorem 1 let us recall (Stein and Weiss [**9**] and Carbery [**2**]) that, for any $\varepsilon > 0$,

$$\|T_\lambda^* f\|_p \leq C\|Mf\|_p + C(\varepsilon)\sum_{k=1}^{\infty} 2^{-k(\lambda-1/2-\varepsilon)}\|S_{2^{-k}}f\|_p,$$

where M is the maximal operator of Hardy and Littlewood and each S_δ is a square function defined as follows:

Consider any $a \in C_0^\infty(\mathbf{R})$ supported in $[-\frac{1}{2}, \frac{1}{2}]$ and satisfying $\|D^\alpha a\|_\infty \leq 1$ for all $0 \leq \alpha \leq 10$. Let $\phi(\xi) = \phi^{(\delta)}(\xi) = a(\delta^{-1}(1-|\xi|))$, $\phi_t(\xi) = \phi(t\xi)$ and

$$S_\delta f(x) = \left(\int_0^\infty |f * \check{\phi}_t|^2(x) t^{-1}\, dt\right)^{1/2}.$$

(To be more precise, each $S_{2^{-k}}$ appearing in the above expression controlling $\|T_\lambda^* f\|_p$ is defined by means of an auxiliary function $a = a_k$ which depends on k.) Define $p_0 = 2(n+1)/(n-1)$ and $r = (\frac{1}{2}p_0)' = (n+1)/2$.

LEMMA 1. *If S_δ is defined as above, then $\|S_\delta f\|_p \leq C(p)\delta^{1-n/2r}\|f\|_p$ for all $2 \leq p < p_0$.*

By summing over all $\delta = 2^{-k}$, $0 \leq k$, we find (since $\frac{1}{2}(n-1)/(n+1) = \frac{1}{2}n/r - \frac{1}{2}$) that, if $2 \leq p < p_0$ and $\lambda > \frac{1}{2}(n-1)/(n+1)$, then T_λ^* is bounded on L^p. The full conclusion of Theorem 1 may then be obtained by interpolating with the more elementary result: T_λ^* is bounded on L^p, for all $1 < p < \infty$, once λ is greater than the critical index $\frac{1}{2}(n-1)$. (For in that case T_λ^* is dominated pointwise by the Hardy-Littlewood maximal operator, since

$$|((1-|\xi|^2)_+^\lambda)^\vee(x)| \leq C(1+|x|)^{-(n+1)/2-\lambda}.)$$

Because of work of Rubio de Francia [**8**] and Carbery [**2**] one expects that the square function S_δ should be controlled by a maximal operator. Let $M_r h = M(h^r)^{1/r}$ for any nonnegative function h; M always denotes the Hardy-Littlewood maximal operator.

LEMMA 2. *For any $0 \leq h$ and $0 < \delta \leq 1$,*

$$\int |S_\delta f|^2(x) h(x)\, dx \leq C\delta^{2-n/r}\int |f|^2(x) M_r h(x)\, dx.$$

Lemma 1 is an immediate corollary. That the controlling operator $h \to M_r h$ should have such a simple form is unexpected; in dimension two the weighted inequality established by Carbery for the Bochner-Riesz multipliers involves averaging over rectangles with eccentricity $\delta^{-1/2}$ and arbitrary orientations.

Our proof relies on an argument given by Stein (see [**6**]), who reduces the problem of estimating the Bochner-Riesz operators to a local one on a fixed cube, and treats the local problem by applying the L^2 restriction theorem. For the local problem one has not only L^{p_0} boundedness, but boundedness of the operator in question from L^2 to L^{p_0}. Our sole innovation is the observation that this stronger local information automatically carries with it a weighted inequality.

To formulate this principle abstractly we suppose that $\mathbf{R}^n$ has been partitioned into a regular lattice of cubes of sidelengths 2^j, and that T is a sublinear operator with the property that, whenever f is supported in a cube Q of the lattice, Tf is supported in a fixed dilate Q^* of Q.

LEMMA 3. *Suppose T is as above, $2 < p_0$ and $r = (\frac{1}{2}p_0)'$. Suppose that for any Q, $\|Tf\|_{p_0} \le A\|f\|_2$ for any f supported in Q. Then for any f defined on $\mathbf{R}^n$ and any testing function $h \ge 0$,*

$$\int |TF|^2(x)h(x)\,dx \le CA^2 2^{jn/r} \int |f|^2(x) M_r h(x)\,dx.$$

PROOF. By the locality assumption it suffices to assume that f is supported in one cube Q of the lattice, and h on Q^*. By Hölder's inequality

$$\begin{aligned}\int |Tf|^2 h &\le \left(\int |Tf|^{p_0}\right)^{2/p_0} \left(\int_{Q^*} h^r\right)^{1/r} \\ &\le CA^2 |Q|^{1/r} \|f\|_2^2 \left(|Q^*|^{-1} \int_{Q^*} h^r\right)^{1/r} \\ &\le CA^2 |Q|^{1/r} \|f\|_2^2 \operatorname*{Inf}_{x\in Q} M_r h(x) \\ &\le CA^2 |Q|^{1/r} \int_Q |f|^2 M_r h.\end{aligned}$$

To deduce Lemma 2, fix a cutoff function $\eta \in C_0^\infty(\mathbf{R}^n)$, identically one on $\{|x| \le 1\}$ and supported on $\{|x| \le 2\}$. Suppose $\delta > 0$ is given, $2^{-i} > \delta \ge 2^{-i-1}$. Let $\varsigma_i(x) = \eta(2^{-i}x)$, and $\varsigma_j(x) = \eta(2^{-j}x) - \eta(2^{1-j}x)$ for all $j > i$.

Apply Lemma 3 to the vector-valued operator $T\colon L^2 \to L^{P_0}(L^2[1/2,4])$ defined by $Tf(x,t) = (f * (\varsigma_j \check{\varphi}_t))(x)$. Let $(\rho,\theta) \in \mathbf{R}^+ \times S^{n-1}$ denote polar coordinates. Since ς_j is a dilate of a fixed Schwartz function, $j \ge i$, and all moments of $\hat{\varsigma}_j$ vanish when $j > i$, routine computation gives

$$|(\hat{\varsigma}_j * \varphi_t)(\rho,\theta)| \le \begin{cases} C_N 2^{i-j}[1 + 2^i|\rho - t|]^{-N}, & \rho \in [1/4, 8], \\ C_N 2^{i-j}\delta^N (1+\rho)^{-N}, & \text{otherwise}\end{cases}$$

for any $N < \infty$ and any $t \in [1/2, 4]$. Set $\hat{f}_0(\rho,\theta) = \hat{f}(\rho,\theta)$ for $\rho \in [1/4, 8]$ and $\hat{f}_0(\rho,\theta) = 0$ otherwise. Let $f_1 = f - f_0$.

We may now show that the parameter A^2 of the lemma is at most $C\delta^2 4^{i-j}$. For any $f \in L^2$,

$$\begin{aligned}\left\| \int_{1/2}^4 |Tf(x,t)|^2\,dt \right\|_{p_0/2} &\le \int_{1/2}^4 \|f * \varsigma_j \check{\varphi}_t\|_{p_0}^2\,dt \\ &\le 2\int_{1/2}^4 \|f_0 * \varsigma_j \check{\varphi}_t\|_{p_0}^2\,dt + 2\int_{1/2}^4 \|f_1 * \varsigma_j \check{\varphi}_t\|_{p_0}^2\,dt.\end{aligned}$$

The first term is the main one. It is

$$\begin{aligned}&2\int_{2^{-1}}^4 \left\| \int_{2^{-2}}^8 \int e^{ix\cdot\rho\theta} \hat{f}(\rho,\theta)(\hat{\varsigma}_j * \varphi_t)(\rho,\theta)\,d\theta\rho^{n-1}\,d\rho \right\|_{p_0}^2 dt \\ &\le C\int_{2^{-1}}^4 \int_{2^{-2}}^8 \left(\left\| \int e^{ix\cdot\rho\theta} \hat{f}(\rho,\theta)(\hat{\varsigma}_j * \varphi_t)(\rho,\theta)\,d\theta \right\|_{p_0} d\rho \right)^2 dt \\ &\le C\int_{2^{-1}}^4 \int_{2^{-2}}^8 \left\| \int e^{ix\cdot\rho\theta} \hat{f}(\rho,\theta)(\hat{\varsigma}_j * \varphi_t)(\rho,\theta)\,d\theta \right\|_{p_0}^2 d\rho[1 + 2^i|\rho - t|]B(t)\,dt\end{aligned}$$

by the Cauchy-Schwarz inequality applied in the variable ρ, where $B(t) = \int[1 + 2^i|\rho - t|^{-1}\,d\rho < C2^{-i} \approx C\delta$. Therefore combining the above bound for $\hat{\varsigma}_j * \varphi_t$ with the L^2 restriction theorem of Tomas and Stein gives the bound

$$C\delta \int_{2^{-1}}^{4} \int_{2^{-2}}^{8} \|\hat{f}(\rho,\cdot)\|^2_{L^2(S^{n-1})} 2^{2(i-j)}[1 + 2^i|\rho - t|]^{1-2}\,d\rho\,dt$$
$$\leq C\delta^2 4^{i-j} \int_{2^{-2}}^{8} \|\hat{f}(\rho,\cdot)\|^2_{L^2(S^{n-1})}\,d\rho = C\delta^2 4^{i-j}\|f\|^2_{L^2}.$$

A straightforward application of Plancherel's theorem and fractional integration, with no appeal to the restriction theorem, yields the same bound for the contribution of f_1.

Therefore by Lemma 3,

$$\left\langle \int_{1/2}^{4} |f * \varsigma_j \check{\varphi}_t|^2\,dt, h \right\rangle \leq C\delta^2 4^{i-j} 2^{jn/r}\langle |f|^2, M_r h\rangle.$$

Now $n/r < 2$, so summing the geometric series over all $j \geq i$ gives

$$\left\langle \int_{1/2}^{4} |f * \check{\varphi}_t|^2\,dt, h \right\rangle \leq C\delta^{2-n/r}\langle |f|^2, M_r h\rangle.$$

In order to conclude the proof via Littlewood-Paley theory we are forced to introduce one more collection of cutoff functions. Select $\rho \in C_0^\infty(\mathbf{R}^n)$, radial and supported in $\{1 \leq |\xi| \leq 3\}$ such that $\sum_{-\infty}^{\infty} \rho(2^k\xi) \equiv 1$ on $\mathbf{R}^n\backslash\{0\}$. ρ_k denotes $\rho(2^k\cdot)$. Then

$$\int_0^\infty |f * \check{\phi}_t|^2 t^{-1}\,dt \leq 3\sum_k \int_0^\infty |(f * \check{\rho}_k) * \check{\phi}_t|^2 t^{-1}\,dt$$
$$= 3\sum_k \int_{2^{k-1}}^{2^{k+2}} |(f * \check{\rho}_k) * \check{\phi}_t|^2 t^{-1}\,dt.$$

By homogeneity and the case $\frac{1}{2} \leq t \leq 4$,

$$\left\langle \int_{2^{k-1}}^{2^{k+2}} |f * \check{\rho}_k * \check{\phi}_t|^2 t^{-1}\,dt, h \right\rangle \leq C\delta^{2-n/r}\langle |f * \rho_k|^2, M_r h\rangle$$

and, therefore,

$$\langle S_\delta f^2, h\rangle \leq C\delta^{2-n/r}\left\langle \sum_k |f * \check{\rho}_k|^2, M_r h \right\rangle \leq C\delta^{2-n/r}\langle |f|^2, M_r h\rangle.$$

The last inequality follows from the weighted norm theory for singular integral operators, since $M_r h$ is an A_1 weight.

PROOF OF THEOREM 2. The necessity of the hypothesis follows from homogeneity considerations and the fact that $\mu\{1 < |\xi| < 2\}$ must be finite. Conversely, if μ satisfies the hypothesis, then it is an immediate corollary of the Tomas-Stein restriction theorem that

$$\left(\int_r^{2r} \left|\hat{f}(\xi)\right|^q d\mu(\xi)\right)^{1/q} \leq CA\|f\|_p \quad \text{for any } 0 < r < \infty.$$

Let $f_k = (\hat{f}\rho_k)\check{}$ where $\{\rho_k\}$ are as above. Suppose $p = 2(n+1)/(n+3)$, so $q = 2$. Then

$$\int \left|\hat{f}(\xi)\right|^2 d\mu(\xi) \le C\sum_k \int \left|\hat{f}_k(\xi)\right|^2 d\mu(\xi) \le CA^2 \sum_k \|f_k\|_p^2.$$

But by Minkowski's inequality and the Littlewood-Paley theory,

$$\sum_k \|f_k\|_p^2 \le \left\|\left(\sum_k |f_k|^2\right)^{1/2}\right\|_p^2 \le C\|f\|_p^2.$$

Since the case $p = 1$, $q = \infty$ is trivial, the case $1 < p < 2(n+1)/(n+3)$ follows by interpolation.

REMARK. This argument is closely related to an almost-orthogonality lemma employed by the author in [**4**].

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