ON ALMOST EVERYWHERE CONVERGENCE
OF BOCHNER-RIESZ MEANS IN HIGHER DIMENSIONS

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ABSTRACT. In \( \mathbb{R}^n \) define \( (T_{\lambda} f)(\xi) = \hat{f}(\xi)(1 - |r^{-1}\xi|^2)^{\lambda} \). If \( n \geq 3 \), \( \lambda > \frac{1}{2}(n-1)/(n+1) \) and \( 2 \leq p < 2n/(n-1-2\lambda) \), then \( \lim_{r \to \infty} T_{\lambda} f(x) = f(x) \) a.e. for all \( f \in L^p(\mathbb{R}^n) \).

The Bochner-Riesz operators in \( \mathbb{R}^n \) are defined as \( (T_{\lambda} f)(\xi) = \hat{f}(\xi)(1 - |\xi|^2)^{\lambda} \), and the associated maximal operators are
\[
T_{\lambda}^* f(x) = \sup_{r>0} |(\hat{f} \cdot (1 - |r\xi|^2))_+^*|(x).
\]

It is conjectured that, when \( \lambda > 0 \), \( T_{\lambda} \) is bounded on \( L^p \) if and only if \( p_0(\lambda) < p < \lambda \), where \( p_0(\lambda) = 2n/(n - 1 - 2\lambda) \). That the restrictions on \( p \) and \( \lambda \) are necessary was shown by Herz [7]. Carleson and Sjölin [3] proved the conjecture in dimension two. Moreover, in \( \mathbb{R}^2 \) Carbery [1] has established boundedness of \( T_{\lambda}^* \) on \( L^p \), and hence almost everywhere convergence of Bochner-Riesz means, for the same range of \( p \) and \( \lambda \) except for the added restriction \( p \geq 2 \). In dimensions greater than two it is known by work of C. Fefferman, Stein and Tomas [5, 6, 10] that \( T_{\lambda} \) is bounded on \( L^p \), provided \( p_0(\lambda) < p < p_0(\lambda) \) and \( \lambda > \frac{1}{2}(n-1)/(n+1) \), but the remaining cases have not been resolved. Our principal result is

**THEOREM 1.** \( T_{\lambda}^* \) is bounded on \( L^p(\mathbb{R}^n) \) whenever \( 2 \leq p < p_0(\lambda) \) and \( \lambda > \frac{1}{2}(n-1)/(n+1) \) for all \( n \geq 3 \).

Interest in \( L^p \) bounds for \( T_{\lambda}^* \) is due to the consequence

**COROLLARY.** If \( f \in L^p(\mathbb{R}^n) \), \( n \geq 3 \), \( \lambda > \frac{1}{2}(n-1)/(n+1) \) and \( 2 \leq p < p_0(\lambda) \), then
\[
\lim_{r \to \infty} (\hat{f} \cdot (1 - |r\xi|^2))_+^*(x) = f(x) \quad \text{a.e.}
\]

The proof is based on the \( L^2 \) restriction theorem of Tomas and Stein [10]. Our second result is a small extension of that theorem, related in spirit to Theorem 1.

**THEOREM 2.** Suppose \( \mu \) is a nonnegative radial measure on \( \mathbb{R}^n \), satisfying \( \mu(\{0\}) = 0 \), and \( n \geq 2 \). Let \( \gamma = n(n-1)/(n+1) \). Suppose \( 1 < p \leq 2(n+1)/(n+3) \) and \( q = ((n-1)/(n+1))^p' \). Then a necessary and sufficient condition that the weighted norm inequality \( \|f\|_{L^q(\mathbb{R}^n, d\mu)} \leq C \|f\|_{L^p(\mathbb{R}^n)} \) hold is that there exist \( A < \infty \) such that, for each \( 0 < r < \infty \), \( \mu \{ |x| < 2r \} \leq Ar^\gamma \).

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To begin the proof of Theorem 1 let us recall (Stein and Weiss [9] and Carbery [2]) that, for any $\varepsilon > 0$,
\[
\|T^{*}_\lambda f\|_p \leq C\|Mf\|_p + C(\varepsilon) \sum_{k=1}^{\infty} 2^{-k(\lambda - 1/2 - \varepsilon)}\|S_{2^{-k}}f\|_p,
\]
where $M$ is the maximal operator of Hardy and Littlewood and each $S_\delta$ is a square function defined as follows:

Consider any $a \in C_0^\infty(\mathbb{R})$ supported in $[-\frac{1}{2}, \frac{1}{2}]$ and satisfying $\|D^\alpha a\|_\infty \leq 1$ for all $0 \leq \alpha \leq 10$. Let $\phi(\xi) = \phi(\delta)(\xi) = a(\delta^{-1}(1 - |\xi|)), \phi_\varepsilon(\xi) = \phi(\varepsilon \xi)$ and
\[
S_\delta f(x) = \left( \int_0^\infty |f * \varphi_t|^2(x) t^{-1} dt \right)^{1/2}.
\]
(To be more precise, each $S_{2^{-k}}$ appearing in the above expression controlling $\|T^{*}_\lambda f\|_p$ is defined by means of an auxiliary function $a = a_k$ which depends on $k$.) Define $p_0 = 2(n + 1)/(n - 1)$ and $r = (\frac{1}{2}p_0)' = (n + 1)/2$.

**Lemma 1.** If $S_\delta$ is defined as above, then $\|S_\delta f\|_p \leq C(p)\delta^{1-n/2r}\|f\|_p$ for all $2 \leq p < p_0$.

By summing over all $\delta = 2^{-k}, 0 \leq k$, we find (since $\frac{1}{2}(n - 1)/(n + 1) = \frac{1}{2}n/r - \frac{1}{2}$) that, if $2 \leq p < p_0$ and $\lambda > \frac{1}{2}(n - 1)/(n + 1), then $T^{*}_\lambda$ is bounded on $L^p$. The full conclusion of Theorem 1 may then be obtained by interpolating with the more elementary result: $T^{*}_\lambda$ is bounded on $L^p$, for all $1 < p < \infty$, once $\lambda$ is greater than the critical index $\frac{1}{2}(n - 1)$. (For in that case $T^{*}_\lambda$ is dominated pointwise by the Hardy-Littlewood maximal operator, since
\[
\|((1 - |x|^2)^{1/2})^{-1}(x)\| \leq C(1 + |x|)^{-(n+1)/2 - \lambda}.
\]

Because of work of Rubio de Francia [8] and Carbery [2] one expects that the square function $S_\delta$ should be controlled by a maximal operator. Let $M_r h = M(h^r)^{1/r}$ for any nonnegative function $h$; $M$ always denotes the Hardy-Littlewood maximal operator.

**Lemma 2.** For any $0 \leq h$ and $0 < \delta \leq 1$,
\[
\int |S_\delta f|^2(x) h(x) dx \leq C \delta^{2-n/r} \int |f|^2(x) M_r h(x) dx.
\]

Lemma 1 is an immediate corollary. That the controlling operator $h \rightarrow M_r h$ should have such a simple form is unexpected; in dimension two the weighted inequality established by Carbery for the Bochner-Riesz multipliers involves averaging over rectangles with eccentricity $\delta^{-1/2}$ and arbitrary orientations.

Our proof relies on an argument given by Stein (see [6]), who reduces the problem of estimating the Bochner-Riesz operators to a local one on a fixed cube, and treats the local problem by applying the $L^2$ restriction theorem. For the local problem one has not only $L^{p_0}$ boundedness, but boundedness of the operator in question from $L^2$ to $L^{p_0}$. Our sole innovation is the observation that this stronger local information automatically carries with it a weighted inequality. To formulate this principle abstractly we suppose that $\mathbb{R}^n$ has been partitioned into a regular lattice of cubes of side lengths $2^j$, and that $T$ is a sublinear operator with the property that, whenever $f$ is supported in a cube $Q$ of the lattice, $Tf$ is supported in a fixed dilate $Q^*$ of $Q$. 
LEMMA 3. Suppose $T$ is as above, $2 < p_0$ and $r = (\frac{1}{2} p_0)'$. Suppose that for any $Q$, $\|TF\|_{p_0} \leq A \|f\|_2$ for any $f$ supported in $Q$. Then for any $f$ defined on $\mathbb{R}^n$ and any testing function $h \geq 0$,

$$\int |TF|^2(x)h(x)\,dx \leq CA^22^{jn/r} \int |f|^2(x)M_r h(x)\,dx.$$  

PROOF. By the locality assumption it suffices to assume that $f$ is supported in one cube $Q$ of the lattice, and $h$ on $Q^*$. By Hölder’s inequality

$$\int |Tf|^2h \leq \left( \int |Tf|^{p_0} \right)^{2/p_0} \left( \int_{Q^*} h^r \right)^{1/r} \leq CA^2|Q|^{1/r} \|f\|_{p_0}^2 \left( |Q^*|^{-1} \int_{Q^*} h^r \right)^{1/r} \leq CA^2|Q|^{1/r} \|f\|_{p_0}^2 \inf_{x \in Q} M_r h(x) \leq CA^2|Q|^{1/r} \int_Q |f|^2 M_r h.$$  

To deduce Lemma 2, fix a cutoff function $\eta \in C_0^{\infty}(\mathbb{R}^n)$, identically one on $\{|x| \leq 1\}$ and supported on $\{|x| \leq 2\}$. Suppose $\delta > 0$ is given, $2^{-i} > \delta > 2^{-i-1}$. Let $\zeta_i(x) = \eta(2^{-i}x)$, and $\zeta_j(x) = \eta(2^{-j}x) - \eta(2^{1-j}x)$ for all $j > i$.

Apply Lemma 3 to the vector-valued operator $T: L^2 \rightarrow L^{p_0}(L^2[1/2,4])$ defined by $Tf(x,t) = (f * (\zeta_j \varphi_t))(x)$. Let $(\rho, \theta) \in \mathbb{R}^+ \times S^{n-1}$ denote polar coordinates. Since $\zeta_j$ is a dilate of a fixed Schwartz function, $j \geq i$, and all moments of $\overset{i}{\varphi}_j$ vanish when $j > i$, routine computation gives

$$|\overset{i}{\varphi}_j * \varphi_t(\rho, \theta)| \leq \begin{cases} C N 2^{i-j} [1 + 2^i |\rho - t|]^{-N}, & \rho \in [1/4, 8], \\ C N 2^{i-j} \delta^N (1 + \rho)^{-N}, & \text{otherwise} \end{cases}$$

for any $N < \infty$ and any $t \in [1/2, 4]$. Set $\overset{0}{f}(\rho, \theta) = \overset{0}{f}(\rho, \theta)$ for $\rho \in [1/4, 8]$ and $\overset{0}{f}(\rho, \theta) = 0$ otherwise. Let $f_1 = f - \overset{0}{f}$.

We may now show that the parameter $A^2$ of the lemma is at most $C \delta^2 4^{i-j}$. For any $f \in L^2$,

$$\left\| \int_{1/2}^4 |Tf(x,t)|^2 \, dt \right\|_{p_0/2} \leq \int_{1/2}^4 \| f * \overset{i}{\varphi}_t \|_{p_0}^2 \, dt \leq 2 \int_{1/2}^4 \| f_0 * \overset{i}{\varphi}_t \|_{p_0}^2 \, dt + 2 \int_{1/2}^4 \| f_1 * \overset{i}{\varphi}_t \|_{p_0}^2 \, dt.$$  

The first term is the main one. It is

$$2 \int_{2^{-1}}^4 \int_{2^{-2}}^8 \left\| \int_{2^{-2}}^8 e^{ix \cdot \rho \theta} \overset{0}{f}(\rho, \theta)(\overset{i}{\varphi}_t)(\rho, \theta) \, d\theta \rho^{n-1} \, d\rho \right\|_{p_0}^2 \, dt \leq C \int_{2^{-1}}^4 \int_{2^{-2}}^8 \left( \left\| \int_{2^{-2}}^8 e^{ix \cdot \rho \theta} \overset{0}{f}(\rho, \theta)(\overset{i}{\varphi}_t)(\rho, \theta) \, d\theta \right\|_{p_0} \right)^2 \, dt \leq C \int_{2^{-1}}^4 \int_{2^{-2}}^8 \left\| \int_{2^{-2}}^8 e^{ix \cdot \rho \theta} \overset{0}{f}(\rho, \theta)(\overset{i}{\varphi}_t)(\rho, \theta) \, d\theta \right\|_{p_0}^2 \, d\rho [1 + 2^i |\rho - t|] B(t) \, dt$$
by the Cauchy-Schwarz inequality applied in the variable $\rho$, where

$$B(t) = \int |1 + 2^i |\rho - t|^{-1} d\rho < C2^{-i} \approx C\delta.$$ Therefore combining the above bound for $\hat{\psi}_j \varphi_t$ with the $L^2$ restriction theorem of Tomas and Stein gives the bound

$$C\delta \int_{2^{-i}}^{2^{i+1}} \| \hat{f}(\rho, \cdot) \|_{L^2(S^{n-1})}^2 [1 + 2^i |\rho - t|]^{1-2} d\rho dt$$

$$\leq C\delta^2 2^{i-j} \int_{2^{i-2}}^{2^i} \| \hat{f}(\rho, \cdot) \|_{L^2(S^{n-1})}^2 d\rho = C\delta^2 2^{i-j} \| f \|_{L^2}^2.$$  

A straightforward application of Plancherel’s theorem and fractional integration, with no appeal to the restriction theorem, yields the same bound for the contribution of $f_1$.

Therefore by Lemma 3,

$$<\int_{1/2}^{2} |f * \psi_j \varphi_t|^2 dt, h> \leq C\delta^2 2^{i-j} 2^{j/2} (|f|^2, M_r h).$$

Now $n/r < 2$, so summing the geometric series over all $j \geq i$ gives

$$<\int_{1/2}^{2} |f * \varphi_t|^2 dt, h> \leq C\delta^2 -n/r (|f|^2, M_r h).$$

In order to conclude the proof via Littlewood-Paley theory we are forced to introduce one more collection of cutoff functions. Select $\rho \in C_0^\infty(R^n)$, radial and supported in $\{1 \leq |\xi| \leq 3\}$ such that $\sum_{-\infty}^{+\infty} \rho(2^k \xi) \equiv 1$ on $R^n \setminus \{0\}$. $\rho_k$ denotes $\rho(2^k \cdot)$.

Then

$$\int_0^\infty |f * \hat{\varphi}_t|^2 t^{-1} dt \leq \sum_k \int_0^\infty [f * \rho_k] * \hat{\varphi}_t|^2 t^{-1} dt$$

$$= \sum_k \int_{2^{k+2}}^{2^{k+2}} |(f * \rho_k) * \hat{\varphi}_t|^2 t^{-1} dt.$$ By homogeneity and the case $\frac{1}{2} \leq t \leq 4$,

$$<\int_{2^{k-1}}^{2^{k+1}} |f * \rho_k * \hat{\varphi}_t|^2 t^{-1} dt, h> \leq C\delta^2 -n/r (|f * \rho_k|^2, M_r h)$$

and, therefore,

$$<Sf^2, h> \leq C\delta^2 -n/r \left( \sum_k |f * \rho_k|^2, M_r h \right) \leq C\delta^2 -n/r (|f|^2, M_r h).$$

The last inequality follows from the weighted norm theory for singular integral operators, since $M_r h$ is an $A_1$ weight.

**Proof of Theorem 2.** The necessity of the hypothesis follows from homogeneity considerations and the fact that $\mu(\{1 < |\xi| < 2\})$ must be finite. Conversely, if $\mu$ satisfies the hypothesis, then it is an immediate corollary of the Tomas-Stein restriction theorem that

$$\left( \int_r^{2r} |\hat{f}(\xi)|^q d\mu(\xi) \right)^{1/q} \leq CA \| f \|_p \quad \text{for any } 0 < r < \infty.$$
Let \( f_k = (\hat{f} \rho_k)^{-} \) where \( \{\rho_k\} \) are as above. Suppose \( p = 2(n + 1)/(n + 3) \), so \( q = 2 \). Then
\[
\int \left| \hat{f}(\xi) \right|^2 d\mu(\xi) \leq C \sum_k \int \left| \hat{f}_k(\xi) \right|^2 d\mu(\xi) \leq C A^2 \sum_k \|f_k\|_p^2.
\]
But by Minkowski's inequality and the Littlewood-Paley theory,
\[
\sum_k \|f_k\|_p^2 \leq \left( \sum_k \|f_k\|^2 \right)^{1/2} \leq C \|f\|_p^2.
\]
Since the case \( p = 1, q = \infty \) is trivial, the case \( 1 < p < 2(n + 1)/(n + 3) \) follows by interpolation.

**Remark.** This argument is closely related to an almost-orthogonality lemma employed by the author in [4].

**References**


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