ON SUCCESSIVE COEFFICIENTS OF UNIVALENT FUNCTIONS

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ABSTRACT. Let \( f(z) \in S \), that is, \( f(z) \) is analytic and univalent in the unit disk \(|z| < 1\), normalized by \( f(0) = f'(0) - 1 = 0 \). Let \( p \) be real and

\[
\{f(z)/z\}^p = 1 + \sum_{n=1}^{\infty} D_n(p)z^n.
\]

Lucas proved that

\[
|D_n(p)| - |D_{n+1}(p)| \leq A n^{(t(p) - 1)/2} \log^{3/2} n, \quad n = 2, 3, \ldots,
\]

for some absolute constant \( A \) and \( t(p) = (2\sqrt{p} - 1)^2 \). In this paper we improve \( t(p) \) as follows:

\[
T(p) = \frac{4p - 1}{2p + t(p)} t(p).
\]

Let the function \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S \), and

\[
F_p(z) = \left\{ \frac{f(z)}{z} \right\}^p = 1 + \sum_{n=1}^{\infty} D_n(p)z^n.
\]

It is a very interesting problem to find a best possible number \( t(p) \) for which the inequality

\[
|D_n(p)| - |D_{n-1}(p)| \leq An^{(t(p) - 1)/2}
\]

holds, where \( A \) is an absolute constant.

This problem was first studied by Goluzin. In 1963 Hayman obtained a precise result \( t(1) = 1 \). In 1956 the author \[2\] proved that \( t(p) = 2p - 1 \) (0 < \( p \) < 1) for \( f \in S^* \). In the general case, the better result \( t(p) = (2\sqrt{p} - 1)^2 \) \((1/4 < p < 1)\) is due to Lucas \[1\].

THEOREM. Let \( f(z) \in S \), and \( F_p(z) = \{f(z)/z\}^p = 1 + \sum_{n=1}^{\infty} D_n(p)z^n \). Then for every \( p \in (1/4, 1] \), we have

\[
|D_n(p)| - |D_{n-1}(p)| \leq An^{(T(p) - 1)/2} \log^{3/2} n, \quad n = 2, 3, \ldots,
\]

where \( A \) is an absolute constant, and

\[
T(p) = \frac{4p - 1}{2p + t(p)} t(p), \quad t(p) = (2\sqrt{p} - 1)^2.
\]

This estimate is obviously better than Lucas's because \((4p - 1)/(2p + t(p)) < 1\) for \(1/4 < p < 1\). Note that if \( p = 1/2 \) then \( zF_p(z^2) \) is an odd univalent function, and so on.
1. **Lemmas.** We require some lemmas. In the following, $A, A_1, A_2, \ldots$ will denote some absolute constants.

**LEMMA 1 [4].** Let $f(z) \in S$. Then

\[
\int_0^{2\pi} \left| \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right|^2 d\theta \leq \frac{A}{1-r} \log \frac{1}{1-r}, \quad \frac{1}{2} \leq r < 1.
\]

**LEMMA 2.** Let $f(z) \in S$ and $F_p(z) = \frac{f(z)}{z} + \sum_{n=1}^{\infty} D_n(p)z^n$. Let $p$ be given such that $\max_{|z|=p} |f(z)| = |f(p)|$. Then for every $p \in (\frac{1}{4}, 1]$ we have

\[
|z - p|^2 |F_p(re^{i\theta})|^2 \leq \begin{cases} 
4r(1 - \rho)^{2p}(1 - r)^{4p}, & 0 < \rho < r < 1, \\
2^{3-2\sqrt{p}}r^{-2-2p} \rho^{2-t(p)}(1 - r)(1 - \rho)^t(p), & 0 < r \leq \rho < 1,
\end{cases}
\]

where $t(p) = (2\sqrt{p} - 1)^2$.

**PROOF.** By Goluzin’s inequality [5]

\[
\left( |1 - z\rho| \left| \frac{z - \rho}{z\rho} \right| \right)^{2x_1x_2} (1 - \rho^2)^x_2^2 (1 - |z|^2)^x_2^2
\]

\[
\leq \left| \frac{1}{f(z)} - \frac{1}{f(p)} \right|^{2x_1x_2} \frac{|z - p|^2 f'(z)}{f^2(z)} \frac{|\rho^2 f'(\rho)|}{f^2(\rho)} |z|^2,
\]

where $x_1, x_2$ are real numbers.

The following inequalities are known:

\[
|s f'(s)/f(s)| \leq (1 + |s|)/(1 - |s|), \quad |s| < 1,
\]

\[
t^{-1}(1 - t)^2|f(t e^{i\theta})| \leq s^{-1}(1 - s)^2|f(s e^{i\theta})|, \quad 0 < s < t < 1.
\]

From inequality (1.5) and the hypothesis that $0 < \rho < r < 1$ and $\max_{|z|=p}|f(z)| = |f(p)|$, it is easy to show

\[
\frac{1}{|f(re^{i\theta})|} + \frac{1}{|f(p)|} \leq \frac{1}{|f(re^{i\theta})|} + \frac{1}{|f(p)|} \leq \frac{r(1 - \rho)^2}{\rho(1 - r)^2 |f(re^{i\theta})|} \leq 2r(1 - \rho)^2.
\]

We choose $x_1 = x_2 = \sqrt{p}$ in (1.3) and notice that $|z - \rho| \leq |1 - \rho z|$ and $|z - \rho| \leq 2r$. Then from (1.4), (1.5), and (1.6) it is not difficult to deduce the first inequality in (1.2).

The second inequality in (1.2) [1] can also be obtained by putting $x_1 = 2\sqrt{p} - 1, \ x_2 = 1$ in (1.3). Thus the proof is complete.

**LEMMA 3.** With the above assumption, we have

\[
J(t) = \int_0^{2\pi} |te^{i\theta} - \rho|^2 \left| \frac{f'(te^{i\theta})}{f(te^{i\theta})} \right|^2 |f(te^{i\theta})|^{2p} d\theta
\]

\[
\leq \begin{cases} 
\frac{A}{(1-t)^{2(1-\rho)^t(p)}} \log \frac{1}{1-\rho}, & \frac{1}{2} < t \leq \rho < 1, \\
\frac{A(1-\rho)^{2p}}{(1-t)^{4p+1}} \log \frac{1}{1-t}, & \frac{1}{2} < \rho \leq t < 1.
\end{cases}
\]

**PROOF.** The proof follows from Lemmas 1 and 2.
LEMMA 4. With the above assumption, write
\[ \varphi(z) = (\rho - z) \left\{ \frac{f(z)}{z} \right\}^p = \rho + \sum_{n=1}^{\infty} (\rho D_n(p) - D_{n-1}(p)) z^n. \]

Then
\[ I(r) = \int_0^{2\pi} |\varphi(re^{i\theta})|^2 d\theta \]
\[ \leq \begin{cases} \frac{A_1}{(1 - \rho)^{t(p)}} \log^2 \frac{1}{1 - \rho}, & \frac{1}{2} \leq r \leq \rho < 1, \\ \frac{A_2}{(1 - \rho)^{t(p)}} + \frac{(1 - \rho)^{2p}}{(1 - r)^{4p-1}} \log^2 \frac{1}{1 - r}, & \frac{1}{2} \leq \rho \leq r \leq 1. \end{cases} \]

PROOF. Since
\[ |z\varphi'(z)|^2 = p^2 \left| (\rho - z) \frac{zf'(z)}{f(z)} \right|^p - (\rho - z) \left\{ \frac{f(z)}{z} \right\}^p - \frac{z}{p} \left\{ \frac{f(z)}{z} \right\}^p \]
\[ \leq A_3 \left( |\rho - z|^2 \left| \frac{zf'(z)}{f(z)} \right|^2 + \left| \frac{f(z)}{z} \right|^{2p} \right) \]
and it is known that
\[ \int_0^{2\pi} \left| \frac{f(te^{i\theta})}{te^{i\theta}} \right|^{4p} d\theta \leq \frac{A_4}{(1 - t)^{4p-1}} \quad (0 < t < 1), \]
for \( p > \frac{1}{4} \), hence
\[ I'(r) = \frac{d}{dr} \int_0^{2\pi} |\varphi(re^{i\theta})|^2 d\theta = \frac{4}{r} \int_0^r \int_0^{2\pi} |\varphi'(te^{i\theta})|^2 t dt d\theta \]
\[ = \frac{4}{r} \left( \int_0^{1/2} + \int_{1/2}^r \right) \int_0^{2\pi} |\varphi'(te^{i\theta})|^2 d\theta t dt \]
\[ \leq A_5 + A_6 \int_{1/2}^r t dt \left( \int_0^{2\pi} |te^{i\theta} - \rho|^2 \left| t \frac{f'(te^{i\theta})}{f(te^{i\theta})} \right|^2 |f_p(te^{i\theta})|^2 d\theta \right) \]
\[ + \int_{1/2}^r \frac{A_7 dt}{(1 - t)^{4p-1}}. \]
If \( \frac{1}{2} < r \leq \rho < 1 \), (1.9), together with the first inequality in (1.7), yields
\[ I'(r) \leq \frac{A_8}{(1 - r)(1 - \rho)^{t(p)}} \log \frac{1}{1 - r}, \quad \frac{1}{2} \leq r < \rho < 1. \]
Integrating both sides of the above inequality with respect to \( r \) from \( \frac{1}{2} \) to \( \rho \) we obtain the first inequality in (1.8).
If \( 1 > r \geq \rho \) we write
\[ \int_0^r dt \int_0^{2\pi} |te^{i\theta} - \rho|^2 \left| t \frac{f'(te^{i\theta})}{f(te^{i\theta})} \right|^2 |f_p(te^{i\theta})|^2 d\theta \]
\[ = \left( \int_0^{1/2} + \int_{1/2}^\rho + \int_\rho^r \right) \int_0^{2\pi} |te^{i\theta} - \rho|^2 \left| t \frac{f'(te^{i\theta})}{f(te^{i\theta})} \right|^2 |f_p(te^{i\theta})|^2 d\theta dt. \]
By Lemma 3, we have

\[ I'(r) \leq A_9 \left( \frac{1}{(1 - r)^{t(p)+1}} + \frac{(1 - \rho)^{2p}}{(1 - r)^{4p-1}} \right) \log \frac{1}{1 - r}, \quad \frac{1}{2} \leq \rho < r < 1. \]

Again integrating both sides of the above inequality from \( \rho \) to \( r \) yields

\[ I(r) \leq I(\rho) + A_10 \left( \frac{r - \rho}{(1 - \rho)^{t(p)+1}} + \frac{(1 - \rho)^{2p}}{(1 - r)^{4p-1}} \right) \log \frac{1}{1 - r}, \quad \frac{1}{2} \leq \rho \leq r < 1. \]

By the first inequality of (1.8), we get the second inequality in (1.8). Thus the lemma follows.

2. Proof of the Theorem. If \( f \in S \), then the rotation \( e^{-i\theta} f(e^{i\theta} z) \) also belongs to \( S \). This rotation does not change the magnitudes of the coefficients of the corresponding function \( F_p(z) \). Thus for a fixed \( \rho \) with \( 0 < \rho < 1 \), there is no loss of generality in supposing that \( \max_{|z|=\rho} |f(z)| = |f(\rho)| \). Write

\[ \varphi(z) = (\rho - z)F_p(z) = \rho + \sum_{n=1}^{\infty} (\rho D_n(p) - D_{n-1}(p)) z^n. \]

By Cauchy’s inequality, we have

\[ n|\rho D_n(p) - D_{n-1}(p)| \leq \frac{1}{2\pi} \int_{0}^{2\pi} \left| \frac{\varphi(re^{i\theta})}{re^{i\theta} - \rho} \right|^p \left| \frac{f(re^{i\theta})}{f(re^{i\theta})} \right|^p d\theta + \int_{0}^{2\pi} \left| \frac{f(re^{i\theta})}{f(re^{i\theta})} \right|^p d\theta \]

\[ \leq \frac{A_{11}}{r^n} \left( \int_{0}^{2\pi} \left| \frac{f(re^{i\theta})}{r}\right|^p d\theta \int_{0}^{2\pi} \left| \frac{f(re^{i\theta})}{r}\right|^p d\theta \right)^{1/2} + \frac{A_{12}}{(1 - r)^{2p-1}}. \]

Application of Lemma 1 and Lemma 4 to the above integration gives

\[ n|\rho D_n(p) - D_{n-1}(p)| \leq \frac{A_{13}}{(1 - r)^{1/2}} \left( \frac{1}{(1 - \rho)^{t(p)}} + \frac{(1 - \rho)^{2p}}{(1 - r)^{4p-1}} \right)^{1/2} \log^{3/2} \frac{1}{1 - r}. \]

Putting \( r = 1 - 1/n, \rho = 1 - n^{-4p-1}/(2p+t(p)) \), we have

\[ |\rho|D_n(p)| - |D_{n-1}(p)|| \leq A_{14} n^{(T(p)-1)/2} \log^{3/2} n. \]

Since \( |D_n(p)| \leq A_{15} n^{2p-1}, \) hence

\[ |D_n(p)| - |D_{n-1}(p)| \leq A_{14} n^{(T(p)-1)/2} \log^{3/2} n + (1 - \rho)|D_{n-1}(p)| \]

\[ \leq A_{16} \left( n^{(T(p)-1)/2} \log^{3/2} n + n^{2p-(4p-1)/(2p+t(p))-1} \right). \]

Let \( x_0 = (4p-1)/(2p+t(p)) \left( \frac{1}{2} < p < 1 \right) \). We see that

\[ 2p - \frac{4p-1}{2p+t(p)} < 2p - x_0 = \frac{1}{2} + \frac{1}{2} x_0 t(p) \]

\[ \leq \frac{1}{2} + \frac{4p-1}{2p+t(p)} \frac{t(p)}{2} = \frac{1}{2} (1 + T(p)). \]

Hence the theorem follows from (2.4).
References


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