ON SUCCESSIVE COEFFICIENTS OF UNIVALENT FUNCTIONS

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ABSTRACT. Let $f(z) \in S$, that is, $f(z)$ is analytic and univalent in the unit disk $|z| < 1$, normalized by $f(0) = f'(0) - 1 = 0$. Let $p$ be real and

$$\{f(z)/z\}^p = 1 + \sum_{n=1}^{\infty} D_n(p)z^n.$$ 

Lucas proved that

$$||D_n(p)\| - \|D_{n+1}(p)\|| \leq An^{(t(p)-1)/2} \log^{3/2} n, \quad n = 2, 3, \ldots,$$

for some absolute constant $A$ and $t(p) = (2\sqrt{p} - 1)^2$. In this paper we improve $t(p)$ as follows:

$$T(p) = \frac{4p - 1}{2p + t(p)} t(p).$$

Let the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S$, and

$$F_p(z) = \left\{\frac{f(z)}{z}\right\}^p = 1 + \sum_{n=1}^{\infty} D_n(p)z^n.$$ 

It is a very interesting problem to find a best possible number $t(p)$ for which the inequality

$$||D_n(p)\| - \|D_{n-1}(p)\|| \leq An^{(t(p)-1)/2}$$

holds, where $A$ is an absolute constant.

This problem was first studied by Goluzin. In 1963 Hayman obtained a precise result $t(1) = 1$. In 1956 the author [2] proved that $t(p) = 2p - 1$ ($0 < p < 1$) for $f \in S^*$. In the general case, the better result $t(p) = (2\sqrt{p} - 1)^2$ ($\frac{1}{4} < p < 1$) is due to Lucas [1].

THEOREM. Let $f(z) \in S$, and $F_p(z) = \{f(z)/z\}^p = 1 + \sum_{n=1}^{\infty} D_n(p)z^n$. Then for every $p \in \left(\frac{1}{4}, 1\right]$, we have

$$||D_n(p)\| - \|D_{n-1}(p)\|| \leq An^{(T(p)-1)/2} \log^{3/2} n, \quad n = 2, 3, \ldots,$$

where $A$ is an absolute constant, and

$$T(p) = \frac{4p - 1}{2p + t(p)} t(p), \quad t(p) = (2\sqrt{p} - 1)^2.$$ 

This estimate is obviously better than Lucas’s because $(4p - 1)/(2p + t(p)) < 1$ for $\frac{1}{4} < p < 1$. Note that if $p = \frac{1}{2}$ then $zF_p(z^2)$ is an odd univalent function, and so on.

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1. **Lemmas.** We require some lemmas. In the following, $A, A_1, A_2, \ldots$ will denote some absolute constants.

**Lemma 1** [4]. Let $f(z) \in S$. Then

$$\int_0^{2\pi} \left| \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right|^2 d\theta \leq \frac{A}{1-r} \log \frac{1}{1-r}, \quad \frac{1}{2} \leq r < 1.$$  

**Lemma 2.** Let $f(z) \in S$ and $F_p(z) = \{f(z)/z\}^p = 1 + \sum_{n=1}^{\infty} D_n(p) z^n$. Let $p$ be given such that $\max_{|z|=\rho} |f(z)| = |f(\rho)|$. Then for every $p \in (\frac{1}{4}, 1]$ we have

$$|f'(re^{i\theta}) - \rho|^2 |F_p(re^{i\theta})|^2 \leq \begin{cases} 4r(1-\rho)^{2p}/(1-r)^{4p}, & 0 < \rho < r < 1, \\ 2^{3-2\sqrt{p}} r^2 - 2p r - t(p)/2/(1-r)(1-\rho)^{t(p)}, & 0 < r \leq \rho < 1, \end{cases}$$

where $t(p) = (2\sqrt{p} - 1)^2$.

**Proof.** By Goluzin's inequality [5]

$$\left| \frac{1 - \rho^2}{z - \rho} \right|^2 (1-\rho^2)x^2 (1-|z|^2)^2 x^2 \leq \begin{vmatrix} 1 \quad 1 \\ f(z) \quad f(\rho) \end{vmatrix}^2 \begin{vmatrix} \frac{1}{f^2(z)} \quad \frac{1}{f^2(\rho)} \end{vmatrix}^2 \frac{|\rho^2 f'(\rho)|}{t(p)} x^2,$$

where $x_1, x_2$ are real numbers.

The following inequalities are known:

$$|f/(s)/(f(s)| \leq (1 + |s|)/(1 - |s|), \quad |s| < 1,$$

(1.5) $t^{-1}(1-t)^2|f(t e^{i\theta})| \leq s^{-1}(1-s)^2|f(s e^{i\theta})|, \quad 0 < s \leq t < 1.$

From inequality (1.5) and the hypothesis that $0 < \rho \leq r < 1$ and $\max_{|z|=\rho} |f(z)| = |f(\rho)|$, it is easy to show

$$\frac{1}{f(re^{i\theta})} + \frac{1}{f(p)} \leq \frac{1}{f(re^{i\theta})} + \frac{1}{f(p)} \leq \frac{1}{f(re^{i\theta})} + \frac{(1-\rho)^2}{\rho(1-r)^2|f(re^{i\theta})|} \leq 2r(1-\rho)^2.$$  \(1.6\)

We choose $x_1 = x_2 = \sqrt{p}$ in (1.3) and notice that $|z - \rho| \leq |1 - \rho^2|$ and $|z - \rho| \leq 2r$. Then from (1.4), (1.5), and (1.6) it is not difficult to deduce the first inequality in (1.2).

The second inequality in (1.2) [1] can also be obtained by putting $x_1 = 2\sqrt{p} - 1$, $x_2 = 1$ in (1.3). Thus the proof is complete.

**Lemma 3.** With the above assumption, we have

$$J(t) = \int_0^{2\pi} |te^{i\theta} - \rho|^2 \left| \frac{f'(te^{i\theta})}{f(te^{i\theta})} \right|^2 \left| f(te^{i\theta}) \right|^{2p} d\theta$$

$$\leq \begin{cases} \frac{A}{(1-t)^2(1-\rho)^{t(p)}} \log \frac{1}{1-r}, & 1/2 < t \leq \rho < 1, \\ \frac{A(1-\rho)^{2p}}{(1-t)^{4p+1}} \log \frac{1}{1-t}, & 1/2 < \rho \leq t < 1. \end{cases}$$

**Proof.** The proof follows from Lemmas 1 and 2.
LEMMA 4. With the above assumption, write

$$\varphi(z) = (\rho - z) \left\{ \frac{f(z)}{z} \right\}^p = \rho + \sum_{n=1}^{\infty} (\rho D_n(p) - D_{n-1}(p)) z^n.$$ 

Then

$$I(r) = \int_0^{2\pi} |\varphi(re^{i\theta})|^2 d\theta$$

(1.8)

$$\leq \begin{cases} 
A_1 \frac{1}{(1-\rho)^{1/p}} \log^2 \frac{1}{1-\rho}, & \frac{1}{2} \leq r \leq \rho < 1, \\
A_2 \left\{ \frac{1}{(1-\rho)^{1/p}} + \frac{(1-\rho)^{2p}}{(1-r)^{4p-1}} \right\} \log^2 \frac{1}{1-r}, & \frac{1}{2} \leq \rho \leq r \leq 1.
\end{cases}$$

PROOF. Since

$$|z\varphi'(z)|^2 = p^2 \left| (\rho - z) z f'(z) \left\{ \frac{f(z)}{z} \right\}^p - (\rho - z) \left\{ \frac{f(z)}{z} \right\}^p - \frac{z}{p} \left\{ \frac{f(z)}{z} \right\}^{2p} \right|^2$$

$$\leq A_3 \left( |\rho - z|^2 \left| \frac{z f'(z)}{f(z)} \right|^2 + \left| \frac{f(z)}{z} \right|^{2p} \right)$$

and it is known that

$$\int_0^{2\pi} \left| \frac{f(te^{i\theta})}{te^{i\theta}} \right|^{4p} d\theta \leq \frac{A_4}{(1-t)^{4p-1}} \quad (0 < t < 1),$$

for $p > \frac{1}{4}$, hence

$$I'(r) = \frac{d}{dr} \int_0^{2\pi} |\varphi(re^{i\theta})|^2 d\theta = \frac{4}{r} \int_0^r \int_0^{2\pi} |\varphi'(te^{i\theta})|^2 t \, dt \, d\theta$$

$$= \frac{4}{r} \left( \int_0^{1/2} + \int_{1/2}^r \right) \int_0^{2\pi} |\varphi'(te^{i\theta})|^2 \, d\theta \, dt$$

(1.9)

$$\leq A_5 + A_6 \int_{1/2}^r t \, dt \left( \int_0^{2\pi} |te^{i\theta} - \rho|^2 \left| \frac{f'(te^{i\theta})}{f(te^{i\theta})} \right|^2 \left| f_p(te^{i\theta}) \right|^2 d\theta \right)$$

$$+ \int_{1/2}^r \frac{A_7 t \, dt}{(1-t)^{4p-1}}.$$ 

If $\frac{1}{2} < r \leq \rho < 1$, (1.9), together with the first inequality in (1.7), yields

$$I'(r) \leq \frac{A_8}{(1-r)(1-\rho)^{1/p}} \log \frac{1}{1-r}, \quad \frac{1}{2} \leq r \leq \rho < 1.$$ 

Integrating both sides of the above inequality with respect to $r$ from $\frac{1}{2}$ to $\rho$ we obtain the first inequality in (1.8).

If $1 > r \geq \rho$ we write

$$\int_0^r dt \int_0^{2\pi} |te^{i\theta} - \rho|^2 \left| \frac{f'(te^{i\theta})}{f(te^{i\theta})} \right|^2 \left| f_p \right|^2 d\theta$$

$$= \left( \int_0^{1/2} + \int_{1/2}^\rho + \int_\rho^r \right) \int_0^{2\pi} |te^{i\theta} - \rho|^2 \left| \frac{f'(te^{i\theta})}{f(te^{i\theta})} \right|^2 \left| f_p \right|^2 d\theta \, dt.$$
By Lemma 3, we have

\[ I'(r) \leq A_9 \left( \frac{1}{(1 - \rho)^{(p+1)}} + \frac{(1 - \rho)^{2p}}{(1 - r)^{4p-1}} \right) \log \frac{1}{1 - r}, \quad \frac{1}{2} \leq \rho < r < 1. \]

Again integrating both sides of the above inequality from \( \rho \) to \( r \) yields

\[ I(r) \leq I(\rho) + A_{10} \left( \frac{r - \rho}{(1 - \rho)^{(p+1)}} + \frac{(1 - \rho)^{2p}}{(1 - r)^{4p-1}} \right) \log \frac{1}{1 - r}, \quad \frac{1}{2} \leq \rho \leq r < 1. \]

By the first inequality of (1.8), we get the second inequality in (1.8). Thus the lemma follows.

2. **Proof of the Theorem.** If \( f \in S \), then the rotation \( e^{-i\rho}f(e^{i\varphi}z) \) also belongs to \( S \). This rotation does not change the magnitudes of the coefficients of the corresponding function \( F_p(z) \). Thus for a fixed \( \rho \) with \( 0 < \rho < 1 \), there is no loss of generality in supposing that \( \max_{|z|=\rho} |f(z)| = |f(\rho)| \). Write

\[ \varphi(z) = (\rho - z)F_p(z) = \rho + \sum_{n=1}^{\infty} (\rho D_n(p) - D_{n-1}(p))z^n. \]

By Cauchy’s inequality, we have

\[ n|\rho D_n(p) - D_{n-1}(p)|| \leq \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{\varphi'(re^{i\varphi})}{r^{n-1}} \right| d\theta \]

\[ \leq \frac{A_{11}}{r^n} \left( \int_0^{2\pi} \left| re^{i\varphi} - \rho \right|^p \left| \frac{f(re^{i\varphi})}{r} \right|^p \left| \frac{f'(re^{i\varphi})}{r} \right|^p d\varphi + \int_0^{2\pi} \left| \frac{f(re^{i\varphi})}{r} \right|^p d\varphi \right) \]

\[ \leq \frac{A_{11}}{r^n} \left( \int_0^{2\pi} \left| re^{i\varphi} - \rho \right|^2 \left| \frac{f(re^{i\varphi})}{r} \right|^{2p} d\varphi \int_0^{2\pi} \left| \frac{f'(re^{i\varphi})}{r} \right|^2 d\varphi \right)^{1/2} + \frac{A_{12}}{(1 - r)^{2p-1}}. \]

Application of Lemma 1 and Lemma 4 to the above integration gives

\[ n|\rho D_n(p)| - |D_{n-1}(p)| \leq A_{13} \left( \frac{1}{(1 - \rho)^{t(p)}} + \frac{(1 - \rho)^{2p}}{(1 - r)^{4p-1}} \right)^{1/2} \log^{3/2} \frac{1}{1 - r}. \]

Putting \( r = 1 - 1/n, \rho = 1 - n^{-(4p-1)/(2p+t(p))} \), we have

\[ |\rho D_n(p)| - |D_{n-1}(p)| \leq A_{14} n^{(T(p)-1)/2} \log^{3/2} n. \]

Since \( |D_n(p)| \leq A_{15} n^{2p-1} \), hence

\[ n|\rho D_n(p)| - |D_{n-1}(p)| \leq A_{14} n^{(T(p)-1)/2} \log^{3/2} n + (1 - \rho)|D_{n-1}(p)| \]

\[ \leq A_{16} \left( n^{(T(p)-1)/2} \log^{3/2} n + n^{2p-(4p-1)/(2p+t(p)-1)} \right). \]

Let \( x_0 = (4p - 1)/(2 + t(p)) \left( \frac{1}{4} < p < 1 \right) \). We see that

\[ 2p - \frac{4p - 1}{2p + t(p)} < 2p - x_0 = \frac{1}{2} + x_0/t(p) \]

\[ \leq \frac{1}{2} + \frac{4p - 1}{2p + t(p)} \frac{t(p)}{2} = \frac{1}{2} (1 + T(p)). \]

Hence the theorem follows from (2.4).
REFERENCES


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