MINIMAL SUPERALGEBRAS
OF WEAK-* DIRICHLET ALGEBRAS

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Abstract. Let $A$ be a weak-* Dirichlet algebra in $L^\infty(m)$ and let $H^\infty(m)$ be the weak-* closure of $A$ in $L^\infty(m)$. It may happen that there are minimal weak-* closed subalgebras of $L^\infty(m)$ that contain $H^\infty(m)$ properly. In this paper it is shown that if there is a minimal, proper, weak-* closed superalgebra of $H^\infty(m)$, then, in fact, that algebra is the unique least element in the lattice of all proper weak-* closed superalgebras of $H^\infty(m)$.

Recall that by definition [6], a weak-* Dirichlet algebra is an algebra $A$ of essentially bounded measurable functions on a probability measure space $(X, \mathcal{A}, m)$ such that (i) the constant functions lie in $A$; (ii) $A + \overline{A}$ is weak-* dense in $L^\infty = L^\infty(m)$ (the bar denotes conjugation); and (iii) for all $f$ and $g$ in $A$,

$$\int_X fg \, dm = \left( \int_X f \, dm \right) \left( \int_X g \, dm \right).$$

The abstract Hardy spaces $H^p = H^p(m)$, $1 \leq p < \infty$, associated with $A$ are defined as follows. For $1 \leq p < \infty$, $H^p$ is the closure of $A$ in $L^p$, while $H^\infty$ is defined to be the weak-* closure of $A$ in $L^\infty$. The space $H^\infty$ is a weak-* closed subalgebra of $L^\infty$.

In recent years the structure of the lattice of proper weak-* closed superalgebras of $H^\infty$ has attracted considerable attention; see, in particular, [2, 3 and 5]. It is easy to construct examples where $\mathcal{L}$ has no least element and no minimal elements. However, in Corollary 5 of [5], we gave a necessary and sufficient condition for $\mathcal{L}$ to have a least element and we characterized it in Corollary 3 of [3]. The question arises: Can $\mathcal{L}$ have minimal elements, but no least element? In this paper we show that the answer is no.

Theorem. If the lattice $\mathcal{L}$ of proper weak-* closed superalgebras of $H^\infty$ has a minimal element, then that element is the least element of $\mathcal{L}$.

Let $B$ be a weak-* closed superalgebra of $A$. Let $B_0 = \{ f \in B; \int_X f \, dm = 0 \}$ and let $I_B$ be the largest weak-* closed ideal of $B$ contained in $B_0$. Then by Lemma 2 of [4], $I_B = \{ f \in L^\infty; \int_X fg \, dm = 0 \text{ for all } g \in B \}$. If $B = H^\infty$, then $I_B = H_0^\infty$. On p. 153 of [4], the measure $m$ is called quasi-multiplicative on $B$ if $\int_X f^2 \, dm = 0$ for every
f \in B \) such that \( \int_E f \, dm = 0 \) for all sets \( E \) such that the characteristic function \( \chi_E \in B \). The measure \( m \) is clearly quasi-multiplicative on \( H^\infty \) and \( L^\infty \). More elaborate examples are given on p. 163 of \([4]\). However, recently Kallenborn and König \([1, \text{Theorem 1.5}]\) showed that \( m \) is always quasi-multiplicative on any weak-* closed superalgebra of \( A \). This fact will play a crucial role in the proof of the theorem.

For any subset \( M \subseteq L^\infty \), \( [M]_L \) will denote the closed linear span of \( M \) in \( L^2 \). If \( E \) is a measurable subset of \( X \), \( \chi_E \) will denote the characteristic function of \( E \). The support set of any function \( f \in L^1 \) will be denoted \( \text{supp}(f) \).

**Lemma 1.** Let \( B \) be a weak-* closed superalgebra of \( A \). If \( f \in B \) and \( \chi_E f \notin I_B \) for every \( \chi_E \in B \) with \( \chi_E f \neq 0 \), then \( \chi_E(f) \in B \).

**Proof.** Set \( M_f = [fB]_L \). Then \( M_f \) is a left-continuous invariant subspace for \( B \); i.e., \( \chi_E M_f \supseteq \chi_E I_B [M_f]_L \) for every nonzero \( \chi_E \in B \) with \( \chi_E M_f \neq \{0\} \). Since the measure \( m \) is quasi-multiplicative on \( B \), by Theorem 1.5 of \([1]\), we may apply Theorem 2 of \([4]\) to conclude that \( M_f = \chi_E q[B]_L \) for some unimodular \( q \) and some \( \chi_E \in B \). Clearly \( \chi_E(f) = \chi_E q \) and so \( \chi_E(f) \in B \).

**Lemma 2.** Suppose that \( B \) is a minimal weak-* closed superalgebra of \( H^\infty \). If \( f \in H^2 \) and \( f \notin [I_B]_L \), then \|f\| > 0 a.e.

**Proof.** Let \( K \) denote the orthogonal complement of \( I_B \) in \( H^2 \). We first show that if \( f \in K \), \( f \neq 0 \), then \( \|f\| > 0 \) a.e. To see this, set \( g = hf \) where \( h \in H^\infty \), \( [hA]_L = H^2 \), and \( |h| = \min\{1, 1/|f|\} \). Then \( g \in B \), and \( \chi_E g \notin I_B \) for every \( \chi_E \in B \) with \( \chi_E g \neq 0 \). Lemma 1 implies that \( \chi_E(f) = \chi_E(g) \) belongs to \( B \). Set \( M_f = [fA]_L \) and \( D = \{g \in B; gM_f \subseteq M_f\} \). Then \( D \) is a weak-* closed superalgebra of \( H^\infty \) with \( D \subseteq B \), and \( \chi_E(f) \in D \). If \( \chi_E(f) \neq 1 \), then \( H^\infty \subseteq D \), and so \( D = B \) since \( B \) is assumed to be minimal. But then, since \( M_f \subseteq H^2_0 = \{g \in H^2; \int_X g \, dm = 0\} \) and \( BM_f \subseteq M_f \), we see that \( M_f \subseteq [I_B]_L \) by Lemma 2 of \([4]\). Thus we conclude that \( f \in [I_B]_L \), contrary to our hypothesis that \( f \in K \). Thus \( \chi_E(f) = 1 \). To complete the proof, choose an arbitrary \( f \in H^2 \) with \( f \notin [I_B]_L \) and write \( f = u + f_0 \) where \( u \in K \) and \( f_0 \in [I_B]_L \). Since \( f \notin [I_B]_L \), \( u \neq 0 \), and so \( |u| > 0 \) a.e., by what we just proved. Again, set \( g = hf \), where \( h \in H^\infty \), \( [hA]_L = H^2 \), and \( |h| = \min\{1, 1/|f|\} \). Then \( g \in B \) and we claim that \( \chi_E g \notin I_B \) for every \( \chi_E \in B \) with \( \chi_E g \neq 0 \). For, if \( \chi_E_0 g \in I_B \), for some \( \chi_E_0 \in B \) with \( \chi_E_0 g \neq 0 \), then \( \chi_E_0 hu \in I_B \). Since the equation \( [hA]_L = H^2 \) implies that \( [hI_B]_L = [I_B]_L \), we find that \( \chi_E_0 u \in I_B \), which contradicts the fact that \( |u| > 0 \) a.e. Lemma 1 now implies that \( \chi_E(f) = \chi_E(g) \) lies in \( B \). So \((1 - \chi_E(f))u = -(1 - \chi_E(f))f_0 \) belongs to \([I_B]_L \cap K = \{0\} \). Thus \((1 - \chi_E(f))u = 0 \) a.e., which implies that \( \chi_E(f) = 1 \) a.e., since \( |u| > 0 \) a.e.

**Proof of the Theorem.** Let \( B \) be a minimal, proper, weak-* closed superalgebra of \( A \), and let \( D \) be any proper weak-* closed superalgebra of \( A \). We must show that \( B \subseteq D \). By Lemma 2 of \([4]\), it suffices to show that \( I_D \subseteq I_B \). Since \( D \supseteq H^\infty \), there is a \( \chi_E \in D \) with \( 0 \leq m(E) \leq 1 \), by Lemma 3 of \([3]\). If \( f \in I_D \), then both \( \chi_E f \) and
(1 - \chi_E)f \in I_D and, in particular, \chi_E f and (1 - \chi_E)f belong to \(H^\infty\). By Lemma 2, both \chi_E f and (1 - \chi_E)f belong to \(I_B\), and so \(f = \chi_E f + (1 - \chi_E)f\) belongs to \(I_B\). Thus \(I_D \subseteq I_B\) and this completes the proof.

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**References**


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