ON RANDOM APPROXIMATIONS AND
A RANDOM FIXED POINT THEOREM
FOR SET VALUED MAPPINGS

V. M. SEHGAL AND S. P. SINGH

ABSTRACT. We prove a random fixed point theorem in a Banach space for set valued mappings and then derive a corollary that yields a fixed point theorem of Bharucha-Reid and Mukherjea, as a special case.

In a recent paper [1], Bharucha-Reid and Mukherjea proved the following stochastic analogue of the well-known Schauder's fixed point theorem.

Theorem 1. Let $S$ be a compact and convex subset of a Banach space $E$ and $T: \Omega \times S \rightarrow S$ be a continuous random operator. Then $T$ admits a random fixed point.

In this paper, we shall show that Theorem 1 can be derived from a more general result. For detailed definitions and terminologies we refer to Bharucha-Reid [1] or a recent paper of Itoh [3]. Throughout, this paper, $(\Omega, \Sigma)$ is a measurable space with $\Sigma$ a sigma algebra of subsets of $\Omega$. The symbol $2^E$ denotes the class of nonempty subsets of a Banach space $E$. A mapping $F: \Omega \rightarrow 2^E$ is called measurable iff for each open set $G$ of $E$,

$$F^{-1}(G) = \{ \omega \in \Omega: F(\omega) \cap G \neq \emptyset \} \in \Sigma.$$ 

It may be pointed out that if $F(\omega)$ is compact for each $\omega \in \Omega$, then $F$ is measurable iff $F^{-1}(C) \in \Sigma$ for each closed subset $C$ in $E$ (see Himmelberg [2] or Itoh [3]).

Let $S$ be a nonempty subset of $E$. Let $T: \Omega \times S \rightarrow 2^E$ be a mapping. $T$ is called

(a) a random operator iff for each fixed $x \in S$, the mapping $T(\cdot, x): \Omega \rightarrow 2^E$ is measurable,

(b) upper (lower) semicontinuous (u.s.c, l.s.c) iff for each fixed $\omega \in \Omega$, $T(\omega, \cdot): S \rightarrow 2^E$ is u.s.c (l.s.c), that is $(T(\omega, \cdot))^{-1}(C)$ is closed (open) subset of $S$ for each closed (open) subset $C$ of $E$,

(c) continuous iff $T$ is both u.s.c and l.s.c.

A single valued measurable mapping $\phi: \Omega \rightarrow E$ is a random fixed point of the random operator $T: \Omega \times S \rightarrow 2^E$ iff $\phi(\omega) \in T(\omega, \phi(\omega))$ for each $\omega \in \Omega$.

The following selection theorem due to Kuratowski and Ryll-Nardzewski [4] is used in the proof of our result.

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Proposition 1. Let $S$ be a separable closed subset of a Banach space $E$ and let $F: \Omega \to 2^S$ be a measurable function such that $F(\omega)$ is compact for each $\omega \in \Omega$. Then there exists a single valued measurable function $\phi: \Omega \to S$ with $\phi(\omega) \in F(\omega)$ for each $\omega \in \Omega$.

For subsets $A$ and $B$ of a normed space, we shall write
\[ d(A, B) = \inf \{ \| x - y \| : x \in A, y \in B \}. \]

The following result is a special case of a well-known result of Reich [5, Lemma 1.6].

Proposition 2. Let $S$ be a compact and convex subset of a Banach space $E$ and $F: S \to 2^E$ be a continuous multifunction such that $F(x)$ is compact and convex for each $x$ in $S$. Then there exists an $x \in S$ with $d(x, Fx) = d(Fx, S)$.

The main result of this paper is

Theorem 2. Let $S$ be a compact and convex subset of $E$ and $T: \Omega \times S \to 2^E$ be a continuous random operator with compact and convex values. Then there exists a single valued measurable map $\phi: \Omega \to S$ satisfying
\[ d(T(\omega, \phi(\omega)), \phi(\omega)) = d(T(\omega, \phi(\omega)), S), \]
for each $\omega \in \Omega$.

We first prove a few lemmas simplifying the proof of Theorem 2.

Lemma 1. Let $S$ be a nonempty subset of a normed space $E$ and $T: S \to 2^E$ be a l.s.c. multifunction. If a sequence $\{x_n\}$ in $S$ converges to an $x_0$ in $S$, then for any $y_0 \in Tx_0$, there exists a subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ and a sequence $\{y_{n_k}\}$ with $y_{n_k} \in T(x_{n_k})$ such that $y_{n_k} \to y_0$.

Proof. Let for each positive integer $n$, $N(y_0, 1/n)$ be a neighborhood of $y_0$ of radius $1/n$. Then $y_0 \in N(y_0, 1/n) \cap Tx_0$. Consequently, for each $n$, there exists an $\varepsilon_n > 0$ such that $Tz \cap N(y_0, 1/n) \neq \emptyset$ for each $z \in N(x_0, \varepsilon_n) \cap S$. Since $x_n \to x_0$, it follows that for each positive integer $k$, there exists $x_{n_k}$, $n_k > n_{k-1}$, such that $T(x_{n_k}) \cap N(y_0, 1/k) \neq \emptyset$. If $y_{n_k} \in T(x_{n_k}) \cap N(y_0, 1/k)$, then $\{y_{n_k}\}$ satisfies the conclusions of Lemma 1.

Lemma 2. Let $S$ be a compact subset of a normed space $E$ and $T: S \to 2^E$ be a continuous multifunction with compact values. Then the real valued functions $f(x) = d(x, Tx)$ and $g(x) = d(Tx, S)$ are continuous on $S$.

Proof. We show that $f$ is continuous. The proof that $g$ is continuous is similar.

Suppose $f$ is not continuous at some point $x_0 \in S$. This implies the existence of an $\varepsilon > 0$ and a subsequence $\{x_n\}$ in $S$ with $x_n \to x_0$ but
\[ |f(x_n) - f(x_0)| > \varepsilon \]
for each $n$. Choose a $y_0 \in Tx_0$ such that $\|y_0 - x_0\| = d(Tx_0, x_0)$. By Lemma 1, there exists a sequence $y_{n_k} \in Tx_{n_k}$ with $y_{n_k} \to y_0$. Thus,
\[ f(x_{n_k}) = d(x_{n_k}, Tx_{n_k}) \leq \|x_{n_k} - y_{n_k}\| \leq \|x_{n_k} - x_0\| + f(x_0) + \|y_0 - y_{n_k}\|. \]
This implies that \( \lim (f(x_n) - f(x_0)) \leq \varepsilon \). Without loss of generality, we may assume that \( \lim (f(x_n) - f(x_0)) \leq \varepsilon \). Now, for each \( n \), choose a \( y_n \in Tx_n \) such that \( d(x_n, Tx_n) = \|x_n - y_n\| \). Since \( T \) is u.s.c, \( TS \) is compact and hence \( \{ y_n \} \) has a subsequence \( \{ y_{n_k} \} \to y_1 \in Tx_0 \) for some \( y_1 \). Consequently,

\[
 f(x_0) = d(x_0, Tx_0) \leq \|x_0 - y_1\| \leq \|x_0 - x_n\| + f(x_n) + \|y_n - y_1\|. 
\]

This yields \( \lim (f(x_0) - f(x_n)) \leq \varepsilon \). Thus \( |f(x_n) - f(x_0)| \leq \varepsilon \) eventually. This contradicts (1). Hence \( f \) is continuous.

**Lemma 3.** Let \( S \) be a nonempty compact subset of a normed vector space \( E \) and \( T: \Omega \times S \to 2^E \) be a multivalued random operator. Then, for each fixed \( x \in S \), the mappings \( g_x \) and \( h_x \) defined by

\[
g_x(\omega) = d(T(\omega, x), x) \quad \text{and} \quad h_x(\omega) = d(T(\omega, x), S)
\]

are measurable.

**Proof.** Let \( \alpha \) be a real. Then it is easy to verify that

\[
\{ \omega \in \Omega: g_x(\omega) < \alpha \} = \{ \omega \in \Omega: T(\omega, x) \cap N(x, \alpha) \neq \emptyset \}.
\]

This implies that \( g_x \) is measurable. To show that \( h_x \) is measurable, let \( D \) be a countable dense subset of \( S \). Then

\[
\{ \omega: h_x(\omega) < \alpha \} = \bigcup_{y \in D} \{ \omega: d(T(\omega, x), y) < \alpha \} = \bigcup_{y \in D} \{ \omega: T(\omega, x) \cap N(y, \alpha) \neq \emptyset \}.
\]

This implies that \( h_x \) is measurable.

**Proof of Theorem 2.** Define a mapping \( F: \Omega \to 2^S \) by

\[
F(\omega) = \{ x \in S: d(T(\omega, x), x) = d(T(\omega, x), S) \}.
\]

Then it follows by Proposition 2 that \( F(\omega) \neq \emptyset \). Further by Lemma 2, \( F(\omega) \) is closed and hence a compact subset of \( S \) for each \( \omega \in \Omega \). We show that \( F \) is measurable. Let \( C \) be a closed subset of \( S \) and \( D \) a countable dense subset of \( S \). For each \( n \), let \( D_n = \{ x \in D: d(x, C) < 1/n \} \) and

\[
C_n = \bigcup_{x \in D_n} \left\{ \omega \in \Omega: d(T(\omega, x), x) < d(T(\omega, x), S) + \frac{1}{n} \right\}.
\]

By Lemma 3, \( C_n \) is measurable for each \( n \). We show that \( F^{-1}(C) = \bigcap_{n=1}^{\infty} C_n \). If \( \omega \in F^{-1}(C) \), then \( F(\omega) \cap C \neq \emptyset \). This implies that there exists an \( x_0 \in C \) with \( d(T(\omega, x_0), x_0) = d(T(\omega, x_0), S) \). Since \( D \) is dense in \( S \), it follows by Lemma 2, that for each fixed \( n \), there exists an \( x_n \in D \) such that \( d(x_n, C) < 1/n \) and

\[
d(T(\omega, x_n), x_n) \leq d(T(\omega, x_0), x_0) + \frac{1}{2n} = d(T(\omega, x_0), S) + \frac{1}{2n}
\]

\[
\leq d(T(\omega, x_n), S) + \frac{1}{n}.
\]
Thus \( \omega \in \bigcap_{n=1}^{\infty} C_n \). Conversely, if \( \omega \in \bigcap_{n=1}^{\infty} C_n \), then for each \( n \), there exists an \( x_n \in D_n \) with \( d(T(\omega, x_n), x_n) < d(T(\omega, x_n)) + 1/n \). Since \( \{x_n\} \subseteq S \) and \( S \) is compact, there exists a subsequence \( x_n \to x_0 \in C \). This implies that \( d(T(\omega, x_0), x_0) = d(T(\omega, x_0), S) \). Thus \( x_0 \in F(\omega) \cap C \), that is, \( \omega \in F^{-1}(C) \). This proves that \( F \) is measurable. Consequently, by Proposition 1 there exists a single valued measurable function \( \phi: \Omega \to S \) with \( \phi(\omega) \in F(\omega) \) for each \( \omega \in \Omega \). This yields \( d(T(\omega, \phi(\omega)), \phi(\omega)) = d(T(\omega, \phi(\omega)), S) \).

The following, a special case of the above result, contains Theorem 1.

**Corollary 1.** Under the hypothesis of Theorem 2, if in addition \( T(\omega, x) \subseteq S \) for each \( \omega \in \Omega, x \in \partial S \) (boundary of \( S \)), then \( \phi \) therein in Theorem 2, is a random fixed point of \( T \).

**Proof.** If for some \( \omega \in \Omega \), \( T(\omega, \phi(\omega)) \cap S = \emptyset \), then \( \phi(\omega) \notin \partial S \). Since \( \phi(\omega) \in S \), it follows that \( \phi(\omega) \) is an interior point of \( S \). Choose a \( y \notin T(\omega, \phi(\omega)) \) such that \( \|y - \phi(\omega)\| = d(T(\omega, \phi(\omega)), \phi(\omega)) \). Since \( y \notin S \), there exist a \( \lambda, 0 < \lambda < 1 \) such that \( (1 - \lambda)y + \lambda\phi(\omega) \in S \). This implies that

\[
d(T(\omega, \phi(\omega)), S) \leq \|y - ((1 - \lambda)y + \lambda\phi(\omega))\|
\]

\[
= \lambda\|y - \phi(\omega)\| < d(T(\omega, \phi(\omega)), S).
\]

This is impossible. Thus, for each \( \omega \in \Omega \), \( T(\omega, \phi(\omega)) \cap S \neq \emptyset \). This implies \( d(T(\omega, \phi(\omega)), S) = 0 \) for each \( \omega \in \Omega \). Hence by Theorem 2, \( d(T(\omega, \phi(\omega)), \phi(\omega)) = 0 \), that is, \( \phi(\omega) \in T(\omega, \phi(\omega)) \) for each \( \omega \in \Omega \).

**References**


**Department of Mathematics, University of Wyoming, Laramie, Wyoming 82070**

**Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John’s, Newfoundland, Canada A1C 5S7**