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AN n-DIMENSIONAL SUBGROUP OF R^{n+1}

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ABSTRACT. A construction given by R. D. Anderson and J. E. Keisler is modified to show that there exists an n-dimensional subgroup G in R^{n+1} such that $\dim G^k = n$ for all k. The group G is connected, locally connected, and divisible.

Introduction. For separable metric spaces the fundamental theorem for the dimension of products is $\dim X \times Y \leq \dim X + \dim Y$. If X is a continuum with $\dim X = n$, then by [**3**], $\dim X \times X = 2n$ or $2n-1$. If X is not a continuum, then it may be that $\dim X \times X = n$ and in fact $\dim X^k = n$ for all k. Anderson and Keisler gave such an example in [**1**] for each n as a subset of R^{n+1}. In this note we show how to modify the construction of that paper to get X to be a subgroup of R^{n+1} and thus a topological group. The example G_n will have the property that G_n meets every nondegenerate subcontinuum in R^{n+1} and consequently will be connected and locally connected. As a topological group it will have a unique completion which will be R^{n+1} since it is densely embedded in R^{n+1} [**2**, Theorem 1, p. 248]. It is also true that the unique topological group completion of G_n^k will be $R^{k(n+1)}$.

The group G_n cannot contain a continuum since a nondegenerate continuum A has the property that $\dim A^k \geq k$. There is no *complete* separable metric space X which is connected and locally connected such that $\dim X^k = \dim X$ for all k unless $\dim X = 0$ or $\dim X = \infty$. The reason for this is that X would either be degenerate or contain an arc.

Notation. Let card(A) be the cardinality of the set A. Let c denote the cardinality of the reals which we think of as an initial ordinal. We denote an s-dimensional hyperplane in R^r by H^s. For $i = 1, 2$, let H_i be a hyperplane of dimension t_i in R^r. Then H_1 and H_2 are in general position with respect to each other if, whenever H_1' and H_2' are translations of H_1 and H_2 with $H_1' \cap H_2' \neq \emptyset$, then $H_1' \cap H_2' = H^t$ where $t = \max\{0, t_1 + t_2 - r\}$.

Let Q denote the rational numbers and $a = (r_1, \ldots, r_s) \in Q^s$. Let H_a be the hyperplane in R^{ns} defined by $H_a = \{(r_1 x, \ldots, r_s x) | x \in R^n\}$. If $a \neq 0$ in Q^s, then H_a is an n-dimensional hyperplane. There are countably many such hyperplanes since Q^s is countable.

The construction. The purpose of this paper is to give a construction proving the following main theorem.

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MAIN THEOREM. *For each positive integer n there is a subgroup G_n in R^{n+1} such that* $\dim G_n = n$ *and* $\dim G_n^k = n$ *for all positive integers k. The group G_n is also connected, locally connected, and divisible.*

The proof is patterned after [**1**]. However, it is as easy to give a complete proof here as to assume familiarity with that proof. We repeat three lemmas from [**1**] without proof.

LEMMA 1. *Let K be a subset of R^n such that $K \cap C \neq \varnothing$ for every nondegenerate continuum C in R^n. Then* $\dim K \geq n-1$.

LEMMA 2. *Given a countable collection of hyperplanes $\{H_i\}_{i=1}^{\infty}$, a k-sphere S, and a hyperplane H, all in R^r, such that $S - H = U_1 \cup U_2$ where $p \in U_1$ with U_i open and closed in $S - H$, and $U_1 \cap U_2 = \varnothing$, then there exists a hyperplane H' such that* (1) $\dim H' = k$, (2) *for each positive integer i, H' is in general position with respect to H_i, and* (3) *$S - H' = V_1 \cup V_2$ where $p \in V_1 \subset U_1$ is open and closed in $S - H'$ and $V_1 \cap V_2 = \varnothing$.*

In R^{ns} choose a countable dense set of points and $(ns-1)$-dimensional spheres S^{ns-1} with rational radius about them such that none of them contains the origin. For each S^{ns-1} choose a countable set of $(ns-1)$-dimensional hyperplanes H^{ns-1} such that their complementary domains form a basis for the topology of S^{ns-1} and such that each H^{ns-1} is in general position with respect to each H_a for all $a \in Q^s$. This is possible by Lemma 2. For each of the countably many S^{ns-1}'s, choose countably many S^{ns-2}'s by $S^{ns-1} \cap H^{ns-1}$ for the H^{ns-1} chosen above.

Inductively, for each $S^{ns-k} = S^{ns-k+1} \cap H^{ns-k+1}$, choose a countable set of hyperplanes H^{ns-k} whose complementary domains in S^{ns-k} form a basis for the topology of S^{ns-k} such that each H^{ns-k} is in general position with respect to each H_a for all $a \in Q^s$. Let $S_i = S_i^{ns-n}$ be the countably many $(ns-n)$-spheres that are obtained when $k = n$.

LEMMA 3. *Let $T \subset R^{ns}$ be such that, for each i, $T \cap S_i = \varnothing$. Then* $\dim T \leq n-1$.

This construction is the same as in [**1**], except that the hyperplanes H^{ns-k} are in general position with respect to a different family of hyperplanes $\{H_a | a \in Q^s\}$ rather than the family γ in [**1**].

Proof of the Main Theorem. We first prove a special case of the Main Theorem. We show that for a fixed s there is a subgroup $G_n \subset R^{n+1}$ such that $\dim G_n = n = \dim G_n^s$. We will then indicate how to modify the proof so that $\dim G_n^s = n$ for all positive integers.

Case 1. For a fixed positive integer s, $\dim G_n^s = n$.

Let $\{C_\alpha | \alpha < c\}$ be an enumeration of the nondegenerate subcontinua in R^{n+1} and assume $0 \in C_0$. We want G_n to be such that $G_n \cap C_\alpha \neq \varnothing$ for all α and $G_n^s \cap Y = \varnothing$, where $Y = \bigcup_{i=1}^{\infty} S_i$ and the S_i's are the $((n+1)s-(n+1))$-spheres in $R^{(n+1)s}$ described just before Lemma 3.

Let $G_0 = \{0\}$. Then suppose that G_β has been chosen for all $\beta < \alpha < c$ with the properties that (1) G_β is a divisible subgroup of R^{n+1}; (2) $G_\beta \subset G_\gamma$ for all $\beta < \gamma < \alpha$; (3) $G_\beta^s \cap Y = \varnothing$ for all $\beta < \alpha$; (4) $G_\beta \cap C_\beta \neq \varnothing$; and (5) $\operatorname{card} G_\beta \leq \aleph_0 \cdot \operatorname{card}([0,\alpha])$ for all $\beta < \alpha$. Then let $G'_a = \bigcup_{\beta<\alpha} G_\beta$. Then G'_α will satisfy (1)–(3) and (5). If $G'_\alpha \cap C_\alpha \neq \emptyset$, then let $G_\alpha = G'_\alpha$ and all five properties are satisfied for $\{G_\beta\}_{\beta<\alpha+1}$. If $G'_\alpha \cap C_\alpha = \emptyset$, then we extend the group G'_α in a manner which we now describe. Let

$$A = \bigcup\{\pi_k(Q \cdot (H_a \cap (S_i + (G'_a)^s))|a \in Q^s,\ i \in N,\ \text{and } k \in \{1, \dots, s\}\}.$$

Note that for each fixed $a \in Q^s, i \in N$, and $k \in \{1, \dots, s\}$, $\pi_k(Q \cdot (H_a \cap (S_i + (G'_\alpha)^s)))$ has cardinality at most $\aleph_0 \cdot \operatorname{card} G'_\alpha$ since $H_a \cap (S_i + (g_1, \dots, g_s))$ is at most two points for all $(g_1, \dots, g_s) \in (G'_\alpha)^s$. This implies that $\operatorname{card} A < c$. This implies that one can choose $P_\alpha \in C_\alpha - A$. Then we let $G_\alpha = G'_\alpha + Q \cdot P_\alpha$. Note that $\operatorname{card} G_\alpha \leq \aleph_0 \cdot \operatorname{card}([0, \alpha+1])$, as required. Suppose that $G^s_\alpha \cap Y \neq \emptyset$. Then there is an $a = (r_1, \dots, r_s) \in Q^s$ and $(g_1, \dots, g_s) \in (G'_\alpha)^s$ such that $(r_1, \dots, r_s)p_\alpha + (g_1, \dots, g_s) \in G^s_\alpha \cap Y$. Clearly, some $r_i \neq 0$ or $(g_1, \dots, g_s) \in Y$ and $G^s_\beta \cap Y \neq \emptyset$ for some $\beta < \alpha$, a contradiction. Now this implies that we have $ap_\alpha \in S_k + (G'_\alpha)^s$ for some k and thus $p_\alpha = \pi_i(ap_\alpha/r_i) \in \pi_i(Q \cdot (H_a \cap (S_k + (G'_\alpha)^s)))$. This implies that $p_\alpha \in A$, a contradiction. Therefore, $G^s_\alpha \cap Y = \emptyset$ and $\{G_\beta\}_{\beta<\alpha+1}$ satisfies (1)–(5). Let $G_n = \bigcup_{\alpha<c} G_\alpha$. Then G_n will be a divisible subgroup of R^{n+1}, $G_n \cap C_\alpha \neq \emptyset$ for all $\alpha < c$, and $G^s_n \cap Y = \emptyset$. Thus $\dim G_n \geq n$ and $\dim G^s_n \leq n$. Thus, $\dim G_n = n = \dim G^s_n$. This proves the special case for a fixed s.

Case 2. Construct G_n such that $\dim G^s_n = n$ for all s.

The construction is similar to Case 1. For each positive integer s, let $Y_s = \bigcup_{i=1}^\infty S_i$ where each S_i is an $[(n+1)s - (n+1)]$-sphere in $R^{(n+1)s}$ as in Case 1. Then we can construct $\{G_\alpha\}_{\alpha<c}$ as in Case 1 with (1′) G_α a divisible subgroup of R^{n+1}; (2′) $G_\alpha \subset G_\beta$ for all $\alpha < \beta < c$; (3′) $G^s_\alpha \cap Y_s = \emptyset$ for all $\alpha < c$ and all positive integers s; (4′) $G_\alpha \cap C_\alpha \neq \emptyset$ for all $\alpha < c$; and (5′) $\operatorname{card} G_\alpha < c$ for all $\alpha < c$. Then $G_n = \bigcup_{\alpha<c} G_\alpha$ will be the required divisible subgroup of R^{n+1}. The strengthening of (3) to (3′) is straightforward and we leave this to the reader.

COROLLARY. *There is a divisible subgroup G_n in R^{n+1} such that* $\dim G_n = \dim G^\omega_n = n$.

PROOF. This follows from Lemma 4 of [**1**], since $\dim G^s_n = n$ for all s.

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