A NOTE ON INTERSECTION OF LOWER SEMICONTINUOUS MULTIFUNCTIONS

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Abstract. Let $F_1$ and $F_2$ be closed and convex valued multifunctions from a topological space $X$ to a normed space $Y$. Assume that the multifunctions are lower semicontinuous at $x_0$. We prove that the intersection multifunction $F = F_1 \cap F_2$ is lower semicontinuous at $x_0$ provided $F(x_0)$ is bounded and has nonempty interior.

1. Introduction. Let $F$ be a multifunction from a topological space $X$ to a uniform space $(Y, \mathcal{U})$, i.e., $F$ is a mapping from $X$ to the family of all subsets of $Y$. $F$ will be called lower semicontinuous (lsc) at $x_0 \in X$ if for every $V \in \mathcal{U}$ there is $U \in N(x_0)$ such that $x \in U$ implies $F(x_0) \subset V(F(x))$, where $N(x_0)$ stands for the neighbourhood filter of $x_0$ and, for $A \subset Y$, $V(A) = \{ y \in Y : (a, y) \in V \text{ for some } a \in A \}$. Such multifunctions will also be called Hausdorff-lower semicontinuous (H-lsc). Accordingly, if $(Y, \| \cdot \|)$ is a normed space then $F$ is lsc at $x_0$ if and only if for every $\varepsilon > 0$ there is $U \in N(x_0)$ such that $F(x_0) \subset F(x) + B_\varepsilon$ for every $x \in U$, where $B_\varepsilon = \{ y \in Y : \| y \| < \varepsilon \}$. Note that $F$ is lsc at $x_0$ if and only if the multifunction $\overline{F}$, i.e., $F(x) = \overline{F}(x)$ for all $x \in X$, is lsc at $x_0$.

Let us recall the usual concept of lower semicontinuity. A multifunction $F$ from a topological space $X$ to a topological space $Y$ is said to be Vietoris-lower semicontinuous (V-lsc) at $x_0 \in X$ if, for every open $G \subset Y$ with $F(x_0) \cap G \neq \emptyset$, there is $U \in N(x_0)$ such that $x \in U$ implies $F(x) \cap G \neq \emptyset$. It is known [6] that $F$ is V-lsc at $x_0$ if and only if it is continuous at $x_0$ as a mapping from $X$ to the hyperspace of all subsets of $Y$ equipped with the lower Vietoris topology. If $Y$ is a uniform space and $F$ is H-lsc at $x_0$ then it is V-lsc at $x_0$. The converse also holds if the set $F(x_0)$ is totally bounded [6].

It is well known that neither H-lsc nor V-lsc are preserved under finite intersections of multifunctions. And, unlike upper semicontinuity [3, 5] no compactness type assumptions are helpful in this context. The classical result of Kuratowski [5, p. 180] says that the multifunction $F = F_1 \cap F_2$ is V-lsc at $x_0$ provided $F_1$ is V-lsc at $x_0$ and $F_2$ is constant, being equal, for every $x \in X$, to a fixed open subset of $Y$. Other results on the intersection of V-lsc multifunctions can be found in [8, 7, 3, and 1].

In this note we provide sufficient conditions for Hausdorff-lower semicontinuity of intersection of multifunctions. Our result improves an earlier result of one of the
authors obtained in [10] for finite-dimensional spaces. The key of the proof is an application of the well-known cancellation law for sets in topological vector spaces ([9], see also [11]): Let $A$, $B$ and $C$ be subsets of a real topological vector space. If $B$ is bounded, and $C$ is nonempty closed and convex, then $A + B \subseteq C + B$ implies $A \subseteq C$.

2. Auxiliary lemmas. In the remaining part of this paper $Y = (Y, || \cdot ||)$ is assumed to be a real normed space.

**Lemma 1.** If $A$ is a convex bounded subset of $Y$ and $\text{int} \ A \neq \emptyset$, then for every $\varepsilon > 0$ there are a set $C \subseteq \text{int} \ A$ and $\delta > 0$ such that $C + B_\delta \subseteq A \subseteq C + B_{\varepsilon}$.

**Proof.** Take an arbitrary $\varepsilon > 0$. Without loss of generality we can assume that $0 \in \text{int} \ A$. Since $A$ is bounded, $\lambda \text{int} \ A \subseteq B_{\varepsilon/2}$ for some $0 < \lambda < 1$. Moreover, there is $\delta > 0$ such that $B_\delta \subseteq \lambda \text{int} \ A$. Thus, putting $C = (1 - \lambda) \text{int} \ A$ we get $C + B_\delta \subseteq A \subseteq C + B_\delta$ because $A = \text{int} \ A$.

The following example shows that the assumption of the boundedness of $A$ cannot be omitted in the above lemma.

**Example.** Let $Y = l^\infty$ and put $A = \{(t_k) \in l^\infty: t_1 \geq 0 \text{ and } t_k \leq k(1 - t_1) \text{ for } k \geq 2\}$. Then $A$ is convex and $\text{int} \ A \neq \emptyset$. Take $\varepsilon = \frac{1}{2}$ and suppose that there are $C \subseteq \text{int} \ A$ and $\delta > 0$ such that

$$C + B_\delta \subseteq A \subseteq C + B_{\varepsilon}.$$ 

For $n \in \mathbb{N}$ let us put $t^n_k = 0$ if $k \neq n$, $k \in \mathbb{N}$ and $t^n_n = n$. Then $x_n = (t_k^n)_{k \in \mathbb{N}} \in A$ for all $n \in \mathbb{N}$, so by $(\ast)$ for every $n \in \mathbb{N}$ there is $y_n = (s_k^n)_{k \in \mathbb{N}} \in C$ such that $||x_n - y_n|| = \sup_{k \in \mathbb{N}}|t_k^n - s_k^n| < \frac{1}{2}$. Take $0 < \alpha < \min\{\delta, \frac{1}{2}\}$. Then $y_n + z \in C + B_\delta \subseteq A$ for every $n \in \mathbb{N}$, where $z = (\alpha, \alpha, \ldots)$. It follows that $s_k^n + \alpha > 0$ and $\alpha + s_k^n \leq n(1 - s_k^n - \alpha)$ for $n \geq 2$, hence $an \leq \frac{1}{2} - \alpha$ for $n \geq 2$, a contradiction.

However, if $Y$ is a finite-dimensional space then Lemma 1 can be strengthened.

**Lemma 2.** Let $A$ be a convex subset of $\mathbb{R}^n$ with nonempty interior. Then for every $\varepsilon > 0$ there are a set $C \subseteq \text{int} \ A$ and $\delta > 0$ such that $C + B_\delta \subseteq A \subseteq C + B_{\varepsilon}$.

**Proof.** Assume that $A$ is unbounded. Otherwise, we can apply Lemma 1. It is clear that the lemma holds if $n = 1$. Suppose then that the thesis of the lemma is satisfied for every convex subset $D \subseteq \mathbb{R}^{n-1}$ with nonempty interior. Take an arbitrary $\varepsilon > 0$ and consider two cases:

1. $A$ contains a line. Without loss of generality we can assume that $A$ contains the $x_n$-axis. Putting $D$ to be the image of $\text{int} \ A$ by the projection into $\mathbb{R}^{n-1}$ we have $\text{int} \ A = D \times \mathbb{R}$. Thus there are $E \subseteq \text{int} \ D$ and $\delta > 0$ such that $E + (B_\delta \cap \mathbb{R}^{n-1}) \subseteq D \subseteq E + (B_\varepsilon \cap \mathbb{R}^{n-1})$. Then denoting by $C$ the set $C = E \times \mathbb{R}$ we get $C \subseteq \text{int} \ A$ and $C + B_\delta \subseteq A \subseteq C + B_{\varepsilon}$.

2. $A$ does not contain a line. We can suppose that $\text{int} \ A$ contains the nonnegative part of the $x_n$-axis and that for some $\lambda_0 > 0$ the set $A_1 = A \cap \{(x, \mu): x \in \mathbb{R}^{n-1}$ and $\mu \leq \lambda_0\}$ is bounded and has nonempty interior. By Lemma 1 there are $G_1 \subseteq \text{int} \ A_1$ and $\alpha > 0$ such that $C_1 + B_\alpha \subseteq A_1 \subseteq C_1 + B_{\varepsilon}$. Let $M$ denote the
hyperplane $M = \{(x, \lambda_0): x \in \mathbb{R}^{n-1}\}$. Since $D = A \cap M$ is a convex body in an $(n-1)$-dimensional space, there are $E \subset \text{int} D$ and $\beta > 0$ such that $E + (B_{\beta} \cap M) \subset D \subset E + (B_{\beta/2} \cap M)$. Put $A_2 = A \cap \{(x, \mu): x \in \mathbb{R}^{n-1} \text{ and } \mu \geq \lambda_0\}$. Then taking $0 < \sigma < \min\{\beta, \epsilon/2\}$ we get the following: For every $y \in \partial A_2$ there exists $z \in \text{int} A_2$, such that $||z - y|| \leq \epsilon/2$ and $z + B_\sigma \subset \text{int} A_2$, where $\partial A_2$ denotes the boundary of $A_2$. Let $C_2$ denote the set $C_2 = \{y \in A_2: \inf ||z - y||: z \in A_2 \geq \sigma\}$. Let us observe that $C_2 + B_\sigma \subset C_2 \subset C_2 + B_\sigma$. Consequently, putting $C = C_1 \cup C_2$ and taking $0 < \delta < \min\{\alpha, \sigma\}$ we get $C + B_\delta \subset A \subset C + B_\delta$.

A multifunction $F$ from $X$ to $Y$ is called locally convex-valued (locally closed-valued) at $x_0 \in X$ if there is $U \subset N(x_0)$ such that $F(x)$ is convex (closed) for all $x \in U$. The following lemma is proved in [10].

**Lemma 3.** Assume that a multifunction $F$ from $X$ to $Y$ is lsc and locally convex-valued at $x_0 \in X$. If $\text{int} F(x_0) \neq \emptyset$ then $\text{int} \{F(x): x \in U\} \neq \emptyset$ for some $U \subset N(x_0)$.

3. Main results.

**Theorem A.** Assume that the multifunctions $F_1$ and $F_2$ from $X$ to $Y$ are locally closed-valued and locally convex-valued at $x_0 \in X$. If $F_1$ and $F_2$ are lsc at $x_0$ and the set $F(x_0) + F_1(x_0) \cap F_2(x_0)$ is bounded and $\text{int} F(x_0) \neq \emptyset$ then the multifunction $F = F_1 \cap F_2$ is lsc at $x_0$.

**Proof.** Let $\epsilon > 0$ be arbitrary. By Lemma 1 there are a subset $C \subset \text{int} F(x_0)$ and $\delta > 0$ such that $C + B_\delta \subset F(x_0) \subset C + B_\sigma$. Since $F_1$ and $F_2$ are lsc at $x_0$, there is $U \subset N(x_0)$ such that $F_i(x_0) \subset F_i(x) + B_\delta$ for all $x \in U$ and $i = 1, 2$. Without loss of generality we can assume that $F_1$ and $F_2$ are closed and convex-valued on $U$. Thus, applying the cancellation law we get $C \subset F(x) = F_1(x) \cap F_2(x)$ for every $x \in U$. But it follows that $F(x_0) \subset C + B_\sigma \subset F(x) + B_\sigma$ for all $x \in U$.

**Theorem B.** Let $Y = \mathbb{R}^n$ and assume that the multifunctions $F_1$ and $F_2$ are locally convex-valued at $x_0 \in X$. If $F_1$ and $F_2$ are lsc at $x_0$ and $\text{int} F(x_0) \neq \emptyset$ then the multifunction $F = F_1 \cap F_2$ is lsc at $x_0$.

**Proof.** Applying Lemma 2 and proceeding as in the proof of Theorem A we obtain that the multifunction $F_1 \cap F_2$ is lsc at $x_0$. Then, by Lemma 3 we find $U \subset N(x_0)$ such that $F_1$ and $F_2$ are convex-valued on $U$ and $\text{int}(F_1(x) \cap F_2(x)) \neq \emptyset$ for all $x \in U$. Then, since $Y$ is finite dimensional, we have $\text{int}(F_1(x) \cap F_2(x)) \neq \emptyset$ and therefore $F_1(x) \cap F_2(x) = F_1(x) \cap F_2(x)$, whenever $x \in U$ (see e.g. [2, p. 253]). Hence, the multifunction $F_1 \cap F_2$, and so also $F$, is lsc at $x_0$.

4. Counterexamples. We give some examples concerning Theorems A and B. The first one shows that the assumption $\text{int} F(x_0) \neq \emptyset$ cannot be omitted.

**Example 1.** Let $Y = \mathbb{R}^2$, $F_1(x) = \text{conv}\{(0, 0), (1, 0), (0, -1)\}$ and $F_2(x) = \text{conv}\{(0, 0), (1, 0), (1, 0)\}$ for all $x \in [0, 1]$. Then $F_1$ and $F_2$ are compact and convex-valued, lsc at every $x \in [0, 1]$ but $F = F_1 \cap F_2$ is not lsc at 0. Note that $F$ is nonempty valued and $\text{int} F(0) = \emptyset$. 

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The second example shows that both multifunctions $F_1$ and $F_2$ must be locally convex-valued.

**Example 2.** Let $Y = \mathbb{R}^2$, $F_1(x) = \text{conv}\{(0, x), (0, 1), (1, 0), (\frac{1}{2}, 0)\} \cup \text{conv}\{(\frac{1}{2}, 0), (1, 0), (1, -1)\}$ and $F_2(x) = \text{conv}\{(0, 0), (1, 0), (1, -1)\}$ for all $x \in [0, 1]$. Then $F_1$ and $F_2$ are compact-valued and lsc at every $x \in [0, 1]$. $F_2$ is convex-valued while $F_1$ is not. $F$ is not lsc at 0.

The third example shows that the boundedness of $F(x_0)$ in Theorem A is essential.

**Example 3.** Let $Y = l^\infty$ and $F_1(x) = \{(t_k) \in l^\infty: t_1 \geq x \text{ and } t_k \leq k - x \text{ for } k \geq 2\}$ and $F_2(x) = \{(t_k) \in l^\infty: t_1 \leq 1 - x \text{ and } t_k \leq k(1 - t_1 - x) \text{ and } t_k \leq k + t_1/k - x/k \text{ for } k \geq 2\}$ for all $x \in [0, 1]$. Then $F_1$ and $F_2$ are closed and convex-valued. Moreover, they are lsc at 0. The set $F(0) = \{(t_k) \in l^\infty: 0 \leq t_1 \leq 1 \text{ and } t_k \leq k(1 - t_1) \text{ for } k \geq 2\}$ has nonempty interior but is not bounded. $F$ is not lsc at 0.

Finally, the last example shows that in all infinite-dimensional normed spaces the multifunctions in Theorem A must be locally closed-valued.

**Example 4.** Let $Y$ be an infinite-dimensional normed space and let $f$ be a linear noncontinuous functional on $Y$. Put $A = \{y \in B_1: f(y) < 0\} \cup \{0\}$ and $B = \{y \in B_1: f(y) > 0\} \cup \{0\}$ where $B_1$ is the closed unit ball of $Y$. Then $A = B = B_1$. Let us define the multifunctions $F_1$ and $F_2$ as follows: $F_1(0) = F_2(0) = B_1$ and $F_1(x) = A$ and $F_2(x) = B$ for all $x \in (0, 1]$. Then $F_1$ and $F_2$ are lsc and convex-valued. The multifunction $F$ is nonempty valued, the set $F(0)$ is bounded with nonempty interior but $F$ is not lsc at 0.

**References**


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