NOTE ON $H^2$ ON PLANAR DOMAINS

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ABSTRACT. An analytic function $f : U \cup U' \to \mathbb{C}$, $U, U' \subset \mathbb{C}$ may belong to $H^2(U)$ and $H^2(U')$ and not to $H^2(U \cup U')$.

Let $U$ be a domain in the complex plane and let $H^2(U)$ be the analytic functions $f$ in $U$ such that $|f|^2$ has a harmonic majorant in $U$. J. Conway asked the following question: Is it possible to have $f \in H^2(U)$, $f \in H^2(U')$ but $f \notin H^2(U \cup U')$? In this note we prove the following

**Theorem.** If $f : D \to \mathbb{C}$ is any analytic function on the unit disk $D$, then $D = U \cup U'$ where $f \in H^2(U)$ and $f \in H^2(U')$.

We set some notation. $D_r = \{z \in \mathbb{C} : |z| < r\}$; $D = D_1$; $h(U, E, z)$ is the harmonic measure for the domain $U$, of the set $E \subset \partial U$ evaluated at $z \in U$. We will consider domains of the following type:

$$U = D_{1/2} \cup \left( \bigcup_{n=1}^{\infty} A_n \right) \cup \left( \bigcup_{n=1}^{\infty} T_n \right)$$

where $A_n$ are annuli $\{z : a_n < |z| < b_n\}$ with $a_1 < \frac{1}{2}$, $a_n < b_n < a_{n+1}$, and $\lim_{n \to \infty} a_n = 1$; the $T_n$ are tubes connecting $A_n$ to $A_{n+1}$, $T_n = \{z : b_n \leq |z| \leq a_{n+1}$ and $|\arg z| < \pi \delta_n\}$. The numbers $a_n, b_n, \delta_n$ will be determined later. We write $U = U_{a_n, b_n, \delta_n}$. We now have two simple observations:

1. $h(U, \partial U \cap \{z : |z| > r\}, 0) \leq \delta_n$ provided $r > b_n$ and $\delta_n > \frac{1}{2}$;
2. $h(U \cap D_{a_n}, \partial(U \cap D_{a_n}) \cap \{z : |z| = a_n\}, 0) \leq \delta_{n-1}$ provided $a_n > \frac{1}{2}$.

To see (1), observe that for Brownian motion started at 0 to exit $U$ through $\partial U \cap \{z : |z| \geq r\}$, it must first get into the tube $T_n$ and the probability of such paths is clearly $\leq \delta_n$. (2) follows in the same way and we leave details to the reader.

**Lemma.** If $\phi : [0, 1] \to \mathbb{R}^+$ is any (strictly) positive decreasing function, we can find $a_n, b_n, \delta_n, a'_n, b'_n, \delta'_n$ so that if $U = U_{a_n, b_n, \delta_n}$ and $U' = U_{a'_n, b'_n, \delta'_n}$, then

1. $D = U \cup U'$,
2. $h(U \cap D_{a_n}, \partial(U \cap D_{a_n}) \cap \{z : |z| \leq r\}, 0) \leq \phi(r)$ provided there is an $m$ such that $\frac{1}{2} < b_m < r$,
3. same as (2) with $U, a_n, b_n$ replaced by $U', a'_n, b'_n$, respectively.

**Proof.** Choose $a_n, b_n, a'_n, b'_n$ so that

$$D = D_{1/2} \cup \left( \bigcup_{n=1}^{\infty} \{z : a_n < |z| < b_n\} \right) \cup \left( \bigcup_{n=1}^{\infty} \{z : a'_n < |z| < b'_n\} \right).$$

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(1) is satisfied no matter what the choice of $\delta_n, \delta'_n$. So choose \( \delta_n = \frac{1}{2}\phi(a_{n+2}) \), \( \delta'_n = \frac{1}{2}\phi'(a_{n+2}) \). If \( r \) is given let \( b_k = \max(b_j : b_j \leq r) \). It follows from observation (1) that
\[
h(U, \partial U \cap \{z : |z| \geq r\}, 0) \leq \delta_k = \frac{1}{2}\phi(a_{k+1}) \leq \frac{1}{2}\phi(r).
\]
From this and the maximum principle,
\[
h(U \cap D_{a_n}, \partial(U \cap D_{a_n}) \cap \partial U \cap \{z : |z| \geq r\}, 0) \leq \frac{1}{2}\phi(r).
\]
Since \( \partial(U \cap D_{a_n}) \cap \partial U \cap \{z : |z| \geq r\} = \partial(U \cap D_{a_n}) \cap \{z : r \leq |z| < a_n\} \) we have
\[
h(U \cap D_{a_n}, \partial(U \cap D_{a_n}) \cap \{z : |z| \geq r\}, 0)
\leq \frac{1}{2}\phi(r) + h(U \cap D_{a_n}, \partial(U \cap D_{a_n}) \cap \{z : |z| = a_n\}, 0).
\]
By our observation, the last term is \( \leq \delta_{n-1} \) and therefore \( \leq \delta_k = \frac{1}{2}\phi(a_{k+2}) \leq \frac{1}{2}\phi(r) \). Putting this together gives (2). (3) is of course similar and the lemma is proved.

**Proof of the Theorem.** If \( f : D \to \mathbb{C} \) is any analytic function (assumed unbounded) choose \( \phi \) as above to satisfy \( \int t\phi(r(t)) \, dt < \infty \), where
\[
r(t) = \min(r : \exists z \text{ with } |z| = r \text{ and } |f(z)| > t).
\]

Construct \( U \) and \( U' \) as in the lemma using this \( \phi \). To see that \( f \in H^2(U) \) consider the obvious harmonic majorant on \( U \cap D_{a_n} \),
\[
g_n(z) = \int_{\partial(U \cap D_{a_n})} |f|^2 \, d\theta(U \cap D_{a_n}, \cdot, z).
\]

If \( \lim_{n \to \infty} g_n \) exists (uniformly on compact sets) then it will be a harmonic majorant for \( |f|^2 \) on \( U \). From Harnack’s theorem, we only need to show \( g_n(0) \) is bounded independently of \( n \). Since
\[
g_n(0) = 2 \int_{E_t} \theta(U \cap D_{a_n}, E_t, 0) \, dt
\]
where \( E_t = \{z \in U \cap D_{a_n} : |f(z)| \geq t\} \subset \{z : |z| \geq r(t)\} \), we have \( h(U \cap D_{a_n}, E_t, 0) \leq \phi(r(t)) \), by the lemma. This and the choice of \( \phi \) imply \( \{g_n(0)\} \) is bounded. The same argument shows that \( f \in H^2(U') \) and the theorem is proved.

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