THE FORELLI PROBLEM CONCERNING IDEALS IN THE DISK ALGEBRA $A(D)$

RAYMOND MORTINI

Abstract. Let $Z(f)$ be the zero set of a function $f \in A(D)$ and $Z(I) = \bigcap_{f \in I} Z(f)$ the zero set of an ideal $I$ in $A(D)$. It is shown that in the disk algebra $A(D)$ every finitely generated ideal $I$ has the weak Forelli property, i.e. there exists a function $f \in I$ such that $Z(f) \cap T = Z(I) \cap T$, where $T$ is the boundary of the unit circle $D$. On the other hand, there exists a finitely generated ideal $I$ in $A(D)$ such that $Z(f) \neq Z(I)$ for each choice of $f \in I$. This provides us with a negative answer to a problem of F. Forelli [1].

1. The disk algebra $A(D)$ is the Banach algebra of all those continuous functions in the closed unit disk $\overline{D}$ which are analytic in the open unit disk $D$, under the usual pointwise algebraic operations and the supremum norm.

Let $Z(f) = \{z \in \overline{D} : f(z) = 0\}$ denote the zero set of a function $f \in A(D)$ and $Z(I) = \bigcap_{f \in I} Z(f)$ the zero set of an ideal $I$ in $A(D)$.

In one of his papers F. Forelli [1] posed the problem of classifying those ideals in $A(D)$ which have the property that there exists a function $f \in I$ such that the zero set of $f$ agrees with the zero set of the ideal $I$, i.e. for which ideals $I$ do we have $Z(f) = Z(I)$ for a function $f \in I$? Such a property will be referred to as the "Forelli property."

It is known that every closed ideal in $A(D)$ has this property. On the other hand, it remained hitherto unsolved whether each finitely generated ideal in $A(D)$ has the Forelli property (Forelli [1, p. 389]). We now solve this problem. The answer, however, will be negative; a fact that was not expected in view of the results in the ring $H(D)$ of all analytic functions in the unit disk $D$.

Definition 1. An ideal $I$ in the disk algebra $A(D)$ has the "Forelli property" if there exists a function $f \in I$ such that $Z(f) = Z(I)$.

Proposition 1. There exists a finitely generated ideal in the disk algebra which does not have the Forelli property.

Proof. As generators we take the functions $f_i(z) = (1 - z)B_i(z)$ ($i = 1, 2$) of v. Renteln [5, p. 139], where $a_n = 1 - n^{-2}$ are the zeros of the Blaschke product $B_1$ and $b_n = a_n + \varepsilon_n$, $\varepsilon_n = n^{-2}\exp[-(1 - a_n)^{-2}] = n^{-2}\exp(-n^4)$, the zeros of the Blaschke product $B_2$. Let $I = (f_1, f_2)$ denote the corresponding ideal. It is now
obvious that $Z(I) = \{1\}$. Hence it is sufficient to show that each function $f \in I$ has infinitely many zeros in $D$.

Let $f = g_1 f_1 + g_2 f_2$ belong to the ideal $I$. By the factorization theorem of F. Riesz, we have $f = Bg$, where $B$ is a Blaschke product and $g$ is a function in $A(D)$ which vanishes nowhere in $D$. Then $B = h_1 f_1 + h_2 f_2$, where $h_1$ and $h_2$ are functions of the Nevanlinna class $N$.

This yields the following estimate (see [5]):

$$|B(a_n)| = |h_2(a_n)| |f_2(a_n)| \leq 2 \exp\left(\frac{C}{1 - |a_n|}\right) \frac{\epsilon_n}{1 - |a_n|} = 2 \exp(Cn^2) \exp(-n^4)$$

where $C > 0$ is a constant that is independent of $n$.

Thus $\lim_{n \to \infty} |B(a_n)| = 0$. This proves that $B$ has infinitely many zeros in $D$, i.e. every member of $I$ vanishes infinitely often in $D$. 

2. Before we proceed, we have to give an auxiliary result of Davie, Gamelin and Garnett.

Let $E$ be an open subset of $T$, the boundary of the unit circle $D$. We denote by $L^\infty_E$ the set of all functions in $L^\infty$ that are continuous in $E$ and by $H^\infty_E$ the set of all bounded analytic functions in $D$ that are continuously extendable to $E$.

**Lemma** (See Garnett [2, P. 399, ex. 15c]). Every function of $L^\infty_E$ can be uniformly approximated by a quotient of the form

$$\frac{\Sigma_{i=1}^{N} c_i B_i}{B}$$

where $B_i (i = 1, \ldots, N)$ and $B$ are Blaschke products in $H^\infty_E$ and $c_i \in \mathbb{C}$.

Our preceding results in §1 show that one cannot expect that even a finitely generated ideal $I$ in the disk algebra has the Forelli property, i.e. that $I$ contains a function $f$ such that $Z(f) = Z(I)$. Therefore, we are going to modify the question of Forelli by considering merely the boundary zero sets. This leads us to the following definition.

**Definition 2.** An ideal $I$ in the disk algebra $A(D)$ has the "weak Forelli property" if there exists a function $f \in I$ such that $Z(f) \cap T = Z(I) \cap T$.

We are now able to prove the main result of this paper.

**Theorem.** Every finitely generated ideal in the disk algebra has the weak Forelli property.

**Proof.** Let $I = (f_1, \ldots, f_N)$ be a finitely generated ideal in $A(D)$. Our goal is the construction of a function

$$f = \sum_{i=1}^{N} H_i f_i \in I$$

such that $\sum_{i=1}^{N} |H_i f_i| \geq C \sum_{i=1}^{N} |f_i|^2$ holds on $T$, where $C > 0$ is a constant. This immediately yields $Z(f) \cap T \subset Z(I) \cap T$, and hence the assertion, because the inclusion $Z(f) \cap T \supset Z(I) \cap T$ holds trivially for each function $f \in I$. 

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
Let \( g_i = f_i^2 \), \( E = T \setminus Z(I) \). The inequality of Cauchy and Schwarz yields the estimate

\[
\left| \sum_{i=1}^{N} \frac{g_i}{\sum_{n=i}^{N} |g_n|} g_i \right| \geq \frac{1}{N} \sum_{i=1}^{N} |g_i| \quad \text{on } E.
\]

Because the sum \( \sum_{i=1}^{N} |g_i| \) does not vanish in \( E \), the functions \( q_i = \frac{g_i}{\sum_{n=i}^{N} |g_n|} \) are in \( L_\infty^E \) (\( i = 1, \ldots, N \)).

We are now going to approximate the functions \( q_i \) as in the Lemma. More precisely, there exist functions \( h_i \in H_\infty^E \) and Blaschke products \( B_i \in H_\infty^E \) such that

\[
\|q_i - h_i/B_i\|_\infty \leq \frac{1}{2N},
\]

where \( \| \cdot \|_\infty \) is the supremum norm in \( L_\infty \). Thus we have by (1) and (2) the following estimates on \( E \):

\[
\left| \sum_{i=1}^{N} \frac{h_i}{B_i} g_i \right| \geq \left| \sum_{i=1}^{N} q_i g_i \right| - \sum_{i=1}^{N} \left| q_i - \frac{h_i}{B_i} \right| |g_i| \geq \left( \frac{1}{N} - \frac{1}{2N} \right) \sum_{i=1}^{N} |g_i|.
\]

To get rid of the denominators on the left side of inequality (3), we multiply with \( |B_1| \cdot \cdots \cdot |B_N| = 1 \). Thus,

\[
\left| \sum_{i=1}^{N} (\varphi_i h_i) g_i \right| \geq \frac{1}{2N} \sum_{i=1}^{N} |g_i| \quad \text{on } E,
\]

where \( \varphi_i = \prod_{j=1, j \neq i}^{N} B_j \) (\( i = 1, \ldots, N \)).

By construction, the functions \( \varphi_i h_i f_i \) belong to \( H_\infty^E \). Because the \( f_i \) are continuous on \( T \) and vanish in \( Z(I) \), the functions \( H_i = \varphi_i h_i f_i \in A(D) \) (\( i = 1, \ldots, N \)). Thus, \( \sum_{i=1}^{N} H_i f_i \in I \) and (4) implies

\[
\left| \sum_{i=1}^{N} H_i f_i \right| \geq \frac{1}{2N} \sum_{i=1}^{N} |f_i|^2 \quad \text{on } T(!)
\]

which was to be proved. □

Remark. Our Theorem does not hold for every given ideal in the disk algebra. There even exist countably generated ideals which do not have the weak Forelli-property, as the following example shows.

Example. Let \( \{z_n\} \) be a sequence of different points in \( T \) such that \( z_n \to 1 \) (\( n \to \infty \)) and \( Z_N = \{z_n; n \geq N\} \cup \{1\} \). Because the \( Z_N \) are closed sets of measure zero, there exist by a theorem of Fatou (Hoffman [4, p. 80]) functions \( f_N \in A(D) \) such that \( Z(f_N) = Z_N \).

Let \( I = (f_1, f_2, \ldots) \) be the ideal generated by the functions \( f_1, f_2, \ldots \) in \( A(D) \). It is now obvious that \( Z(I) \cap T = \{1\} \). But, on the other hand, each function of the ideal \( I \) has the form \( f = \sum_{i=1}^{N} g_i f_i \) for an integer \( N \) and suitable chosen functions \( g_i \in A(D) \). This implies that \( Z_N \subseteq Z(f) \). Thus every function \( f \in I \) has infinitely many zeros on \( T \). Consequently, there exists no function \( f \in I \) such that \( Z(f) \cap T = Z(I) \cap T \). □
With regard to the Forelli problem, we are also going to investigate the prime ideals in the disk algebra. Whereas the maximal ideals in $A(D)$ satisfy the Forelli property, there exist, on the other hand, prime ideals which do not have even the weak Forelli property.

**Proposition 2.** In the disk algebra there exist prime ideals which do not have the weak Forelli property.

**Proof.** First we define the set $M = \{ f \in A(D): z = 1 \text{ is not a cluster point of the zeros of} \ f \text{ on} \ T \}$.

It is obvious that $M$ is closed under multiplication. Let $I = (f_1, f_2, \ldots)$ be the ideal of the preceding example. By construction, the ideal $I$ does not intersect $M$. By the Lemma of Zorn there exists an ideal $P \supset I$ that is maximal relative to $P \cap M = \emptyset$. This ideal is now prime (see [3, p. 6]). Moreover, the zero set $Z(P) = \{1\}$.

But for every function $f \in P$, the point $z = 1$ is a cluster point of the zeros of $f$ on $T$, because $P \cap M = \emptyset$. Thus there exists no function in $P$ such that $Z(f) \cap T = Z(P) \cap T$. □

This paper is a part of the author's doctoral thesis at the University of Karlsruhe (Germany).

**References**


Mathematisches Institut I, Universität Karlsruhe, Karlsruhe, Germany