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SOME SHARP WEAK-TYPE INEQUALITIES FOR HOLOMORPHIC FUNCTIONS ON THE UNIT BALL OF $\mathbf{C}^n$

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ABSTRACT. Let $B^n = \{z \in \mathbf{C}^n : |z| < 1\}$, $S^n = \partial B^n$ and let σ_n be the Haar measure on S^n. Then for all $f \in H^p$ $(1 \leqslant p < \infty)$ such that $\operatorname{Im}(f(0)) = 0$ and $t > 0$,

$$\sigma_n(\{z \in S^n : |f(z)| \geqslant t\}) \leqslant C_p \cdot \frac{\|\operatorname{Re} f\|_p^p}{t^p}$$

for some constant C_p depending only on p. The best constant C_p is found for $1 \leqslant p \leqslant 2$.

Let $\mathbf{C}^n$ be an n-dimensional complex space with norm

$$\|z\| = \left(|z_1|^2 + \cdots + |z_n|^2\right)^{1/2}$$

and unit ball $B^n = \{z \in \mathbf{C}^n : \|z\| < 1\}$. By σ_n we shall denote the rotation-invariant, normalized Borel measure on $S^n = \partial B^n$. We shall write D and T instead of B^1 and S^1. For a σ_n-measurable function $f: S^n \to \mathbf{C}$ and $p \geqslant 1$, let us define

$$\|f\|_p = \left(\int_{S^n} |f|^p \, d\sigma_n\right)^{1/p}.$$

If $\| f \|_p < \infty$ and if the Poisson integral $\mathrm{P}[f]$ of the function f (see [**5**, p. 41]) is a holomorphic function, then we shall write $f \in H^p(S^n)$. Kolmogorov proved [**4**] that there exists a constant $C > 0$ such that, if $f \in H^1(T)$ and if $f(0) = \mathrm{P}[f](0)$ is real, then

$$\sigma_1(\{z \in T : |\operatorname{Im} f(z)| \geqslant t\}) \leqslant C \cdot \frac{\|\operatorname{Re} f\|_1}{t}, \tag{1}$$

for all $t > 0$. In other words, the operator $\operatorname{Re} f \to \operatorname{Im} f$ is of the weak type 1-1 (see [**7**]). The best constant of inequality (1) was found by Davis [**3**]. Baernstein [**2**] gave an elementary proof of inequality (1) with the best constant. His proof was modified by Tomaszewski [**6**], who found the best constant in a weak-type inequality for the operator $\operatorname{Re} f \to f$. In this paper we shall prove similar sharp weak-type inequalities for spaces $H^p(S^n)$.

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THEOREM. *If* $1 \leqslant p \leqslant 2, f \in H^p(S^n)$ *and* $\operatorname{Im} f(0) = 0$, *then for all* $t > 0$,

$$\sigma_n\big(\{z \in S^n : |f(z)| \geqslant t\}\big) \leqslant C_p \cdot \frac{\|\operatorname{Re} f\|_p^p}{t^p}, \tag{2}$$

where

$$C_p = \frac{\sqrt{\pi}}{2} \cdot \frac{p \cdot \Gamma(p/2)}{\Gamma((p+1)/2)}.$$

The constant C_p *is the best possible in this inequality.*

We shall need the following

LEMMA. *Let* u_p, *for* $1 \leqslant p \leqslant 2$, *be the Poisson integral of the function* $\gamma_p(e^{it}) = |\cos t|^p$ *defined on* T. *The inequalities*

(i) $u_p(z) \leqslant u_p(0) + |\operatorname{Re} z|^p$,

(ii) $u_p(0) \leqslant u_p(x)$

hold for $z \in D$ *and* $-1 \leqslant x \leqslant 1$.

PROOF. Let $u_p(z) = \operatorname{Re}(\sum_{k=0}^{\infty} a_k \cdot z^k)$ for some real numbers a_k. It is easy to see that $a_{2n+1} = 0$ for $n = 0, 1, 2, \ldots$. We shall prove that

$$(-1)^n \cdot a_{2n} \leqslant 0, \tag{3}$$

for $n = 1, 2, \ldots$. We have

$$\begin{aligned}
(-1)^n \cdot a_{2n} &= \frac{(-1)^n}{\pi} \cdot \int_{-\pi}^{\pi} |\cos t|^p \cdot \cos 2nt\, dt \\
&= \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} |\sin t|^p \cdot \cos 2nt\, dt \\
&= \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} (1 - \cos^2 t)^{p/2} \cdot \cos 2nt\, dt \\
&= \frac{1}{\pi} \cdot \sum_{m=1}^{\infty} b_m \cdot \int_{-\pi}^{\pi} \cos^{2m} t \cdot \cos 2nt\, dt = \frac{1}{\pi} \cdot \sum_{m=1}^{\infty} b_m \cdot I_{2m,2n},
\end{aligned}$$

where $b_m < 0$ are real numbers such that $(1 - s)^{p/2} = 1 + \sum_{m=1}^{\infty} b_m \cdot s^m$ and $I_{m,n} = \int_{-\pi}^{\pi} \cos^n t \cdot \cos mt\, dt$. Since $I_{m,n} = \frac{1}{2} \cdot (I_{m-1,n-1} + I_{m-1,n+1})$ it can be easily proved (by induction on m) that $I_{m,n} \geqslant 0$. This ends the proof of inequality (3). Now, let us note that for $z = x + iy \in D$,

$$\begin{aligned}
&u_p(z) - u_p(0) - a_2 \cdot (x^2 - y^2) \\
&\quad = \operatorname{Re}\left(\sum_{m=2}^{\infty} a_{2m} \cdot z^{2m}\right) \leqslant (x^2 + y^2) \cdot \sum_{m=2}^{\infty} |a_{2m}| \\
&\quad = (x^2 + y^2) \cdot \big(u_p(0) - u_p(i) - a_2\big) = (x^2 + y^2) \cdot \frac{2 - p}{2p} \cdot a_2,
\end{aligned}$$

since

$$a_0 = \frac{2}{\pi} \cdot \frac{\Gamma((p+1)/2) \cdot \Gamma(1/2)}{p \cdot \Gamma(p/2)},$$

$$a_2 = \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} |\cos t|^{p+2}\, dt - 2a_0 = \frac{2p}{p+2} \cdot a_0.$$

But $(2-p)/2p \leqslant 1$ and

$$x^2 \cdot ((2-p)/2p) \cdot a_2 + x^2 \cdot a_2 = a_0 \cdot x^2 \leqslant x^2 \leqslant x^p.$$

Hence, inequality (i) follows. Let us turn to (ii). For $0 \leqslant \alpha \leqslant \pi/2$ let v_α be the Poisson integral of the characteristic function, defined on T, of the set $\{z \in T\colon |\operatorname{Im} z| \geqslant \sin\alpha\}$. Then

$$v_\alpha(z) = \frac{1}{\pi} \cdot \operatorname{Arg}\left(\frac{1 - ia - z - ia}{1 - ia + z + ia}\right) + \frac{1}{\pi} \cdot \operatorname{Arg}\left(\frac{1 + iaz + z - ia}{1 + iaz - z + ia}\right) + 1,$$

where $a = (\cos\alpha)/(1 + \sin\alpha)$. We see that

$$v_\alpha(x) = \frac{2}{\pi} \cdot \operatorname{Arg}\left[1 - a^2 - ia\left(\frac{1+x}{1-x} + \frac{1-x}{1+x}\right)\right] + 1 \leqslant v_\alpha(0).$$

But $u_p(z) = 1 - \int_0^{\pi/2} v_\alpha(z)\, d\mu(\alpha)$ for some positive measure μ ($d\mu(t) = p \cdot \cos^{p-1} t \cdot \sin t\, dt$). Thus, inequality (ii) follows.

Proof of the Theorem. We shall first prove that equality holds in (2) for inner functions $f\colon S^n \to \mathbf{C}$ such that $f(0) = 0$ (i.e. functions $f \in H^p(S^n)$ such that $|f(z)| = 1$ a.e. on S^n and $\int_{S^n} f(z)\, d\sigma_n(z) = 0$). The existence of such functions was proved by Aleksandrov [**1**]. If f is inner and $f(0) = 0$, then

$$\int_{S^n} h(f(z))\, d\sigma_n(z) = \int_T h(z)\, d\sigma_1(z),$$

for every continuous function h defined on T (see [**5**, p. 405]). Taking $h(z) = |\operatorname{Re} z|^p$, we see (2) cannot hold with any constant smaller than the constant C_p defined above.

We shall prove inequality (2) for the case $n = 1$. For each function $f \in H^p(S^n)$, we have

$$\|\operatorname{Re} f\|_p^p = \int_{S^n} \int_T |\operatorname{Re} f(\xi z)|^p\, d\sigma_1(\xi)\, d\sigma_n(z).$$

If we now apply the statement of the Theorem, for the case $n = 1$, to the integrals $\int_T |\operatorname{Re} f(\xi z)|^p\, d\sigma_1(\xi)$, we shall get the general case. Thus, let us assume that $f \in H^p(T)$. Let $\varphi\colon \mathbf{C} \to R$ be a function such that $\varphi(z) = |\operatorname{Re} z|^p$ for $z \notin D$ and $\varphi(z) = u_p(z)$ for $z \in D$, where u_p is defined in the lemma. The function φ is continuous and since the function $|\operatorname{Re} z|^p$ is subharmonic, we have $\varphi(z) \geqslant |\operatorname{Re} z|^p$ for $z \in \mathbf{C}$. It follows that φ is subharmonic on $\mathbf{C}$. Let $E = \{z \in T\colon |f(z)| \geqslant 1\}$ and let us define the functions $\omega(z) = \mathrm{P}[\chi_{T\setminus E}](z)$, $h(z) = \mathrm{P}[|\operatorname{Re} f|^p](z)$, where $\chi_{T\setminus E}$ is a characteristic function of the set E and P denotes the Poisson integral for the unit disk D. We shall prove that

$$\varphi(f(z)) \leqslant h(z) + \varphi(0) \cdot \omega(z), \tag{4}$$

where we write $f(z) = \mathrm{P}[f](z)$. It suffices to check this inequality for $z \in T$, since the function $\varphi \circ f$ is subharmonic and the function $h(z) + \varphi(0) \cdot \omega(z)$ is harmonic. If $z \in T - E$, then $f(z) \in D$. Hence, for this case (4) follows from (i) and the definition of the function h and the function ω. If $z \in E$, then $\varphi(f(z)) = |\mathrm{Re}\, f(z)|^p = h(z)$, and (4) also holds for this case, hence for every $z \in T$. Taking $z = 0$ in (4) and applying (ii), we get

$$\begin{aligned}\varphi(0) &\leqslant \varphi(f(0)) \leqslant h(0) + \varphi(0) \cdot \omega(0)\\ &= \|\mathrm{Re}\, F\|_p + \varphi(0) \cdot \sigma_1(T - E).\end{aligned}$$

This ends the proof of inequality (2) for the case $t = 1$, since $\sigma_1(T - E) = 1 - \sigma_1(E)$ and $\varphi(0) = u_p(0) = (C_p)^{-1}$. A general case can be proved by considering the function f/t instead of f.

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