A p-LOCAL SPLITTING OF $BU(n)$
KENSHI ISHIGURO

ABSTRACT. Let $p$ be a prime and let $n > 1$. A necessary and sufficient condition that the classifying space $BU(n)$ is $p$-equivalent to the product of nontrivial spaces is that $p$ does not divide $n$.

Let $U(n)$ denote the Lie group of unitary $n \times n$ matrices, and let $U = \varprojlim U(n)$. In this paper we study the classifying space $BU(n)$ and determine those primes at which this space is equivalent to a product. The result is quite different from the infinite case. Recall that when we pass to the limit there are two types of splitting that occur. The first requires no localization;

$$BU \simeq BT^1 \times BSU.$$ 

The proof of this splitting is elementary, of course, but it does use the $H$-structure on $BU$. The second type of splitting is truly $p$-primary. At each prime $p$, $BU$ splits into a product of $p$ irreducible spaces

$$BU \simeq_p \prod_{n=1}^{p} B(2n,p).$$

This was first proved by Peterson [6]. A thorough account of this splitting is also given in Zabrodsky's book [8].

The main result of this paper is

**THEOREM.** If $1 < n < \infty$, then $BU(n)$ is irreducible at $p$ if and only if $p$ divides $n$. If $p$ does not divide $n$, then

$$BU(n) \simeq_p BT^1 \times BSU(n)$$

and both factors are irreducible.

Most of the work in our proof involves showing that when $p$ divides $n$, the unstable algebra $H^*(BU(n); F_p)$ is indecomposable over the Steenrod algebra. In other words, it cannot be expressed as the tensor product of two nontrivial unstable $A^*$-algebras. Here $A^*$ denotes the Steenrod algebra modulo the two-sided ideal generated by the Bockstein coboundary. Our proof uses reflection groups and the methods and results of Adams and Wilkerson [2].

I would like to thank my advisor, C. W. Wilkerson, for his help and encouragement.

1. $A^*$-algebras. Let $H^*$ and $E^*$ be $A^*$-algebras. We say that $E^*$ is a retract of $H^*$ if there are $A^*$-maps

$$E^* \xrightarrow{i} H^*$$

such that $\pi \cdot i = 1_{E^*}$.
**Proposition 1.** Suppose that $H^* \cong H^*(BT^n : F_p)^W$ where $W$ is a suitable group of $A^*$-automorphisms. Then any $A^*$-retract of $H^*$ is likewise the ring of invariants in $H^*(BT^k : F_p)$ for some integer $k$ and some group $W'$.

**Proof.** The argument uses the main result of Adams and Wilkerson [2] and the naturality of $A^*$-maps. Suppose that $E^*$ is a retract of $H^*$. Obviously, $E^*$ is embedded in $H^*(BT^n : F_p)$. Since $E^* = \pi H^*$, it is Noetherian. So it remains to show that $E^*$ satisfies the two conditions in [2, Theorem 1.2]:

(i) $E^*$ is integrally closed in its field of fractions.

(ii) If $y \in E^{2dp}$ and $Q^r y = 0$ for any $r \geq 1$, then $y = \alpha^p$ for some $\alpha \in E^*$.

First, suppose $\alpha \in q(E^*)$. Here $q(R)$ denotes the quotient field of an integral domain $R$. Let $i$ be the monomorphism of the fields $q(E^*) \rightarrow q(H^*)$ such that $i|_{E^*} = i$. If $\alpha$ is integral over $E^*$, then the image $\tilde{i}(\alpha)$ is integral over $H^*$. Since $H^*$ is integrally closed, $\tilde{i}(\alpha)$ lies in $H^*$. Let us write $\alpha = \alpha_1 / \alpha_2$ where $\alpha_1 \in E^*$. Thus, we get $i(\alpha_1) = i(\alpha_2) \cdot \beta_0$ for some $\beta_0 \in H^*$. Applying the map $\pi$, it follows that

$$\pi \cdot i(\alpha_1) = \pi \cdot i(\alpha_2) \cdot \pi(\beta_0), \quad \alpha_1 = \alpha_2 \cdot \pi(\beta_0).$$

Since $\pi(\beta_0) \in E^*$, we conclude that $\alpha$ lies in $E^*$. So $E^*$ is integrally closed. Next, suppose $y \in E^{2dp}$ and $Q^r y = 0$ for any $r \geq 1$. Since $i$ is an $A^*$-map, then $Q^r i(y) = 0$. Thus there is $x \in H^{2d}$ such that $i(y) = x^p$.

Once again we apply the map $\pi$, getting $\pi i(y) = \pi(x)^p, y = \pi(x)^p$ where $\pi(x) \in E^{2d}$. This completes the proof.

2. Generalized reflection groups. Let $V$ be a finite-dimensional vector space over a field $k$. A pseudo-reflection of $V$ is a linear automorphism $w$ such that $\text{rank}(1 - w) = 1$. We say that a vector $u$ is a direction of a pseudo-reflection if it is an eigenvector for the eigenvalue that is not equal to 1.

Let $p: G \rightarrow \text{GL}(V)$ be a linear representation. A nonzero vector $t \in V$ is called $G$-invariant if $p(g)t = t$ for any $g \in G$. The representation $p$ is called reducible with respect to a $G$-invariant vector $t$ if there is a hyperplane $V_0$ in $V$ such that $V = V_0 \oplus \langle t \rangle$ and, for any $g \in G$, the automorphism $p(g)$ has the form $\gamma \oplus 1$ for some $\gamma \in \text{GL}(V_0)$.

**Proposition 2.** Let $W$ be the group generated by pseudo-reflections $w_1, \ldots, w_r$. Assume that each $w_i$ is a direction of $w_i$ and that $t$ is $W$-invariant. Then $W$ is reducible with respect to $t$ if and only if the vector $t$ does not belong to the subspace spanned by $u_1, \ldots, u_r$.

**Proof.** Suppose that $W$ is reducible and that $V_0$ is the hyperplane. Let $w$ be one of the generators $w_1, \ldots, w_r$ and let $u$ be a direction of $w$. We can write $u = v_0 + bt$ for some $v_0 \in V_0$ and $b \in k$. If $a$ is the eigenvalue which is not 1, it follows that

$$0 = w(u) - au = w(v_0 + bt) - a(v_0 + bt) = w(v_0) - av_0 + b(1 - a)t.$$

Since $w(v_0) \in V_0$, we get $b(1 - a) = 0$. So $b = 0$ and $u = v_0 \in V_0$. This shows that $\text{Span}(u_1, \ldots, u_r) \subset V_0$. Therefore, $t$ does not belong to $\text{Span}(u_1, \ldots, u_r)$.

Conversely, if $t \notin \text{Span}(u_1, \ldots, u_r)$, then there is a hyperplane $V_0$ such that $\text{Span}(u_1, \ldots, u_r) \subset V_0$ and $V = V_0 \oplus \langle t \rangle$. Given a generator $w$ with direction $u$, we
have a decomposition; \( V = \langle u \rangle \oplus \ker(w-1) \). Let us write \( V_w = V_0 \cap \ker(w-1) \). We claim \( V_0 = \langle u \rangle \oplus V_w \). In fact, we see that \( \ker(w-1) = V_w \oplus U \) for some subspace \( U \).

Since \( V_0 \cap U = 0 \), we get \( V_0 \oplus U \subset V \) so that \( \dim U \leq 1 \) and hence \( \dim V_w \geq n - 2 \). We notice that \( V_w \neq \ker(w-1) \) since \( t \notin V_0 \). Therefore, \( \dim V_w = n - 2 \). We now see that \( w \cdot V_0 \subset V_0 \) since \( w((u)) \subset \langle u \rangle \) and \( w \) fixes \( V_w \) pointwise. Thus \( V_0 \) is invariant under the \( W \)-action and hence \( W \) is reducible with respect to \( t \). This completes the proof.

3. **Proof of the Theorem.** First assume that \( p \) divides \( n \). By Borel [3, Proposition 29.2], we see that \( H^*(BU(n) : F_p) = H^*(BT^n : F_p) S_n \) where \( S_n \) is the symmetric group. Suppose that \( H^*(BU(n) : F_p) \) is \( A^* \)-decomposable. According to Proposition 1, there is an \( A^* \)-isomorphism \( \theta \) from \( H^*(BU(n) : F_p) \) to \( H^*(BT^{n_1} : F_p)^{W_1} \otimes H^*(BT^{n_2} : F_p)^{W_2} \) for some integers \( n_1 \) and \( n_2 \) and some suitable groups \( W_1 \) and \( W_2 \) because each \( A^* \)-algebra is a retract. By Adams and Wilkerson [2, Proposition 1.10], we can find an \( A^* \)-map \( \phi \) which makes the following diagram commutative:

\[
\begin{array}{ccc}
H^*(BT^n : F_p) & \xrightarrow{\phi} & H^*(BT^{n_1+n_2} : F_p) \\
\uparrow & & \uparrow \\
H^*(BU(n) : F_p) & \xrightarrow{\theta} & H^*(BT^{n_1} : F_p)^{W_1} \otimes H^*(BT^{n_2} : F_p)^{W_2}.
\end{array}
\]

In this diagram the vertical maps are injective. If \( W = W_1 \times W_2 \), then clearly

\[
H^*(BT^{n_1} : F_p)^{W_1} \otimes H^*(BT^{n_2} : F_p)^{W_2} = H^*(BT^{n_1+n_2} : F_p)^W.
\]

Recall that \( H^*(BU(n) : F_p) \) is a polynomial ring in \( n \) variables. Thus the maximum number of elements in \( H^*(BT^{n_1+n_2} : F_p)^W \) which can be algebraically independent over \( F_p \) is \( n \); so we have \( n_1 + n_2 = n \).

Recall that \( H^*(BU(n) : F_p) \hookrightarrow H^*(BT^n : F_p) \) is a Galois extension with Galois group \( S_n \). Lang [5, p. 247] shows that for any \( w \in W \) there exists \( \sigma \in S_n \) such that \( w \phi = \phi \sigma \). We claim that \( \phi \) is invertible. In fact, if an \( A^* \)-map \( \psi \) covers \( \theta^{-1} \), then \( \psi \cdot \phi \) covers \( \theta^{-1} \cdot \theta = \text{identity} \); so the map \( \psi \cdot \phi \) differs from the identity map by a permutation. Thus \( \phi \) is invertible and hence bijective for dimensional reason. Consequently \( \sigma = \phi^{-1} \cdot w \phi \). Thus it follows that, if \( H^*(BU(n) : F_p) \) is \( A^* \)-decomposable, then the group \( S_n \) is conjugate to \( W_1 \times W_2 \) in \( \text{GL}(n : F_p) \). It is well known that the symmetric group is not the product of two nontrivial subgroups. Consequently one of the \( W_i \)'s must be trivial and it follows that this representation of \( S_n \) is reducible with respect to an \( S_n \)-invariant vector.

Regard \( H^2(BT^n : F_p) \) as a vector space over \( F_p \) with basis \( t_1, \ldots, t_n \). The symmetric group acts on this vector space by the rule \( \sigma(t_i) = t_{\sigma(i)} \). Recall that \( S_n \) is generated by the transpositions \( \sigma_1, \ldots, \sigma_{n-1} \) where \( \sigma_i = (i, i+1) \) and that the vector \( t = \sum_{i=1}^n t_i \) is \( S_n \)-invariant.

Suppose \( p \) is odd. Each \( \sigma_i \) is a pseudo-reflection and the vector \( u_i = t_i - t_{i+1} \) is a direction. Since the representation of \( S_n \) is reducible with respect to \( t \), Proposition 2 shows that the \( S_n \)-invariant vector \( t \) does not belong to \( \text{Span}(u_1, \ldots, u_{n-1}) \). Thus \( \{u_1, \ldots, u_{n-1}, t\} \) must be a basis. Equivalently the following \( n \times n \) matrix must be
nonsingular:
\[
\begin{pmatrix}
1 & 1 \\
-1 & 0 \\
\vdots & \vdots \\
0 & \ddots \\
\end{pmatrix}
\]
Since the determinant of the matrix is $n$, the prime $p$ does not divide $n$. This contradicts our assumption.

In the case $p = 2$, it is enough to show that there is no such hyperplane $V_0$ when $n$ is even. We recall that $V$ has basis $t_1, \ldots, t_n$. Suppose that $V_0$ exists. Since $S_n$ acts on $V_0$, without loss of generality we may assume that $t_1 + \cdots + t_m$ is contained in $V_0$ for some $m < n$. If $m = 1$, then we can find $\sigma \in S_n$ such that $t_i = \sigma t_1$. Thus each $t_i$ belongs to $V_0$. But $\dim V_0 = n - 1$, thus $m > 1$. If $m = 2$, then for each $k = 2, \ldots, n$ we can find permutations $\tau_1, \ldots, \tau_{k-1}$ such that
\[
t_1 + t_k = \sum_{r=1}^{k-1} (t_r + t_{r+1}) = \sum_{r=1}^{k-1} \tau_r (t_1 + t_2).
\]
Thus, each $t_1 + t_k \in V_0$ and hence $t = \sum_{k=2}^{n} (t_1 + t_k)$ is contained in $V_0$ since $n$ is even. This contradicts the assumption $V = V_0 \oplus (t)$. Therefore, $2 < m < n$. Then we have, however, that
\[
t_m + t_{m+1} = t_1 + \cdots + t_m + \sigma_m (t_1 + \cdots + t_m) \in V_0
\]
and therefore $t_1 + t_2 \in V_0$. This is also a contradiction. We now conclude that $V_0$ does not exist.

Next assume that $p$ does not divide $n$. Consider the map $f: T^1 \times SU(n) \to U(n)$ given by
\[
f(z, A) = \begin{pmatrix}
z \\
\vdots \\
z
\end{pmatrix} \cdot A
\]
where $z \in T^1$ and $A \in SU(n)$. It is easy to see that this map is a homomorphism with fibre $\mathbb{Z}/n$. On the level of classifying spaces, this map induces another fibration
\[
B\mathbb{Z}/n \to BT^1 \times BSU(n) \xrightarrow{Bf} BU(n).
\]
Localization preserves fibrations; consequently, when this fibration is localized at $p$, the fibre $B\mathbb{Z}/n$ becomes contractible since $p$ does not divide $n$. Hence the map $Bf$ becomes a homotopy equivalence.

It remains to show that $BSU(n)$ and $BT^1$ are irreducible at $p$. For $BT^1$, this is obvious because $BT^1(p) = K(\mathbb{Z}_{(p)}, 2)$. For $BSU(n)$, the argument is very similar to the one used before. Namely, if $BSU(n)$ split as a product at the prime $p$, then it would follow that the representation of its Weyl group $S_n$ in $GL(n-1; \mathbb{F}_p)$ would be conjugate to a product. Just as before, it would follow that this representation would, in fact, be reducible with respect to a nonzero $S_n$-invariant vector $t'$. But such a vector would correspond to a generator of $H^2(BSU(n); \mathbb{F}_p) = 0$. This contradiction completes the proof of the Theorem.
A p-LOCAL SPLITTING OF $BU(n)$

REFERENCES


DEPARTMENT OF MATHEMATICS, WAYNE STATE UNIVERSITY, DETROIT, MICHIGAN 48202