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CONTINUITY OF THE INVERSE

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ABSTRACT. We present a simple device for proving the continuity of the inverse in a group with a locally Čech-complete topology which makes the multiplication continuous; our proof even works in case the topology is regular and locally strongly countably complete.

We denote by X a group with Hausdorff topology[1] such that the multiplication $X \times X \to X$ is continuous. The following propositions hold:

(A) *If X is locally compact, then X is a topological group* [**2**].

(B) *If X is completely metrizable, then X is a topological group* [**7**].

(C) *If X is locally Čech-complete, then X is a topological group* [**1**].

It is well known that locally compact and completely metrizable spaces are Čech-complete ([**4**, 3.9.1 and 4.3.26], respectively), so Proposition (C) simultaneously generalizes Propositions (A) and (B). We shall give new and short proofs of these propositions.

Let e be the neutral element of X, $\mathfrak{U}$ the filter of e-neighborhoods; X is a topological group, if for every $U \in \mathfrak{U}$ there exists $V \in \mathfrak{U}$ with $V^{-1} \subset U$. Our proofs are based on the following

LEMMA. *Let X be regular.*

(a) *For $U \in \mathfrak{U}$ there exists a sequence $(U_n)_{n\in\mathbf{N}}$ in $\mathfrak{U}$ such that*

$$\overline{U}_1^2 \subset U, \quad \overline{U}_{n+1}^2 \subset U_n \quad \textit{for } n \in \mathbf{N}.$$

(b) *If $(U_n)_{n\in\mathbf{N}}$ is chosen as in* (a), *$x_n \in U_n$, $y_k := x_1 \cdots x_k$, and $(y_k)_{k\in\mathbf{N}}$ has a clusterpoint, then for every $n \in \mathbf{N}$ there exists $k > n$ such that $x_k^{-1} \in U_n$.*

PROOF (a) This clearly follows from the continuity of the multiplication in (e, e) and the regularity of X.

(b) Let y be a clusterpoint of $(y_k)_{k\in\mathbf{N}}$, $n \in \mathbf{N}$. Since yU_{n+1} is a neighborhood of y, there exists $k > n+1$ such that $y_{k-1} \in yU_{n+1}$; and then

$$x_k^{-1} = y_k^{-1} y_{k-1} \in y_k^{-1} y U_{n+1}. \tag{1}$$

But $y_k^{-1}y$ is a clusterpoint of $(y_k^{-1}y_j)_{j\in\mathbf{N}}$, and for $j > k$ we have

$$y_k^{-1} y_j = x_{k+1} \cdots x_j \in U_{k+1} \cdots U_j \subset U_k,$$

which implies

$$y_k^{-1} y \in \overline{U}_k \subset U_{k-1}. \tag{2}$$

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[1]As to notation from topology we keep to [**4**].

From (1) and (2) we get $x_k^{-1} \in U_{k-1}U_{n+1} \subset U_{n+1}^2 \subset U_n$.

Now the proofs of the propositions. For $U \in \mathfrak{U}$ we chose a sequence $(U_n)_{n\in\mathbf{N}}$ as in the lemma and show by contradiction that there is an $n \in \mathbf{N}$ with $U_n^{-1} \subset U$.

PROOF OF (A). We may assume that U is compact. Supposing that $U_n^{-1} \not\subset U$ for every $n \in \mathbf{N}$ we chose $x_n \in U_n$ with $x_n^{-1} \notin U$. Since $y_k = x_1 \cdots x_k \in U_1 \cdots U_k \subset U$ for every $k \in \mathbf{N}$, and U is countably compact, the sequence $(y_k)_{k\in\mathbf{N}}$ has a clusterpoint; so $x_n^{-1} \in U$ for some $n \in \mathbf{N}$ by the lemma, a contradiction.

PROOF OF (B). Let the topology of X be given by the complete metric d; we may assume that $(U_n)_{n\in\mathbf{N}}$ is a basis of $\mathfrak{U}$. Supposing that $U_n^{-1} \not\subset U$ for every $n \in \mathbf{N}$ we chose $x_n \in U_n$ with $x_n^{-1} \notin U$. Then $x_n \to e$ and so $xx_n \to x$ for every $x \in X$. Beginning with $\overline{x}_1 := x_1$ we thus inductively find a subsequence $(\overline{x}_n)_{n\in\mathbf{N}}$ of $(x_n)_{n\in\mathbf{N}}$ such that for $\overline{y}_n := \overline{x}_1 \cdots \overline{x}_k$ we have $d(\overline{y}_k, \overline{y}_{k+1}) < 2^{-k}$ for $k \in \mathbf{N}$. Then $(y_k)_{k\in\mathbf{N}}$ is convergent. Since $\overline{x}_n \in U_n$, we get $\overline{x}_n^{-1} \in U$ for some $n \in \mathbf{N}$ from the lemma, a contradiction.

PROOF OF (C). We may assume that U is closed in X and Čech-complete with respect to the induced topology [**4**, 3.9.6]. There exists a sequence $(\mathcal{A}_n)_{n\in\mathbf{N}}$ of relatively open covers of U such that any family of closed subsets of U, which has the finite intersection property and contains sets of diameter less that $\mathcal{A}_n$ for every $n \in \mathbf{N}$, has nonempty intersection [**4**, 3.9.2]. We may assume that U_n is open and has diameter less than $\mathcal{A}_n$.

(3) Every sequence$(x_n)_{n\in\mathbf{N}}$ with $x_n \in U_n$ has a clusterpoint in $K := \bigcap_{n=1}^{\infty} U_n$.

Indeed, $F_n := \{x_k : k > n\} \subset U_{n+1}$; since $\overline{F}_n \subset U_n$ and U_n has diameter less than $\mathcal{A}_n$, we obtain $\emptyset \neq \bigcap_{n=1}^{\infty} \overline{F}_n \subset K$.

(4) K is a subgroup of X.

Indeed, clearly $K^2 \subset K$; we have to show that $K^{-1} \subset K$. For $x \in K$ we have $x_n := x \in U_n$, $n \in \mathbf{N}$, and $y_k = x_1 \cdots x_k = x^k \in K \subset U_k$; from (3) we know that $(y_k)_{k\in\mathbf{N}}$ has a clusterpoint. From our lemma we get $x_k^{-1} \in U_n$ for every $n \in \mathbf{N}$ and some $k > n$, i.e. $x^{-1} \in K$.

Supposing now that $U_n^{-1} \not\subset U$ for every $n \in \mathbf{N}$ we chose $x_n \in U_n$ with $x_n^{-1} \notin U$. From (3) we get a clusterpoint $a \in K$ of $(x_n)_{n\in\mathbf{N}}$. Then e is a clusterpoint of $(z_n)_{n\in\mathbf{N}}$, $z_n := a^{-1}x_{x+1}$; thus every $x \in X$ is a clusterpoint of $(xz_n)_{n\in\mathbf{N}}$; from (4) it is clear that $z_n \in KU_{n+1} \subset U_n$. Beginning with $\overline{z}_1 := z_1$ and $V_1 := U_1$ we inductively find a subsequence $(\overline{z}_n)_{n\in\mathbf{N}}$ of $(z_n)_{n\in\mathbf{N}}$ and a sequence $(V_n)_{n\in\mathbf{N}}$ of open sets such that

(5) $y_k := \overline{z}_1 \cdots \overline{z}_k \in V_k, V_k$ has diameter less than $\mathcal{A}_k$, and $\overline{V}_{k+1} \subset V_k$ for every $k \in \mathbf{N}$.

From (5) we obtain a clusterpoint of $(y_k)_{k\in\mathbf{N}}$ in the same way as in the proof of (3). Since $\overline{z}_n \in U_n$, we get $\overline{z}_k^{-1} \in U_1$ for some $k \in \mathbf{N}$ from the lemma. But it is $\overline{z}_k = a^{-1}x_{m+1}$ for some $m \geqq k$, and $a^{-1} \in K$ by (4), and thus $x_{m+1}^{-1} = \overline{z}_k^{-1}a^{-1} \in U_1K \subset U_1^2 \subset U$, a contradiction.

REMARKS. (a) It is well known that in a group with a locally compact or completely metrizable topology which makes the multiplication separately continuous the multiplication is continuous ([**3**, **6**], respectively); it seems to be unknown whether Čech-completeness is sufficient too.

(b) In our proof of Proposition (C) we only used that X is regular and locally strongly countably complete in the sense of [**5**]; the proof of Brand does not work in this case.

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