ON G-SYSTEMS AND G-GRADED RINGS

P. GRZESZCZUK

Abstract. Rings graded by finite groups and homomorphic images of such rings are studied. Obtained results concern finiteness conditions and radicals.

Introduction. Our aim in this paper is the study of rings graded by finite groups. To obtain some results we need information on homomorphic image of a graded ring (cf. Theorem 5). The proofs of other results work in this more general situation as well. For these reasons the paper concerns G-systems, defined as follows.

Let G be a finite group with identity e. A ring R is said to be G-system if $R = \sum_{g \in G} R_g$, where $R_g$ are such additive subgroups of R that $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. If for all $g, h \in G, R_g R_h = R_{gh}$, $R$ is called [3, 4, 7] the Clifford system.

Certainly any G-graded ring is a G-system and the class of G-systems is homomorphically closed, while G-graded rings do not necessarily have this property. It is easy to check that every G-system is a homomorphic image of a G-graded ring.

In this paper we prove that a “Clifford type” theorem holds for every G-system R. Namely, we show that simple $\mathbb{A}$-modules ($\mathbb{A}$-bimodules) are completely reducible $\mathbb{E}$-modules ($\mathbb{E}$-bimodules) and that $\mathbf{j}(R, \mathbb{A}) = \mathbf{j}(R) \cap R_\mathbb{A}$, $\mathbf{U}(R, \mathbb{E}) \subseteq U(R)$, where $\mathbf{j}(\cd), U(\cd)$ denote the Jacobson and the Brown-McCoy radical, respectively. Using this method we obtain, in particular, a quite different proof from those given by M. Cohen and S. Montgomery [2] and M. Van den Bergh [6] of Bergman’s conjecture. We also prove that the $\mathbb{E}$-module $M$ is Noetherian if and only if $M$ is Noetherian as an $\mathbb{E}$-module.

1. We start with

Theorem 1. If the G-system $R = \sum_{g \in G} R_g$ has unity 1, then $1 \in R_\mathbb{E}$.

Proof. Define for any nonempty subset $S$ of $G$, $R_S = \sum_{s \in S} R_s$. It is clear that for all $\emptyset \neq S, T \subseteq G, R_S R_T \subseteq R_{ST}$. We prove by induction on $|G \setminus S|$ that if $e \in S$ then $1 \in R_S$. If $S = G$ then $1 \in R_G = R_\mathbb{E}$. Assume the result is true for subsets of cardinality $|S|$. Let $x \in G \setminus S$; then $|S \cup \{x\}| > |S|$ and $|x^{-1}S \cup \{e\}| > |S|$. Hence by induction assumption $1 \in R_S + R_x$ and $1 \in R_{x^{-1}S} + R_\mathbb{E}$. That is, there exist $\alpha(S) \in R_S$ and $\alpha(x) \in R_x$ so that $1 + \alpha(S) = \alpha(x)$ and $\beta(x^{-1}S) \in R_{x^{-1}S}$, $\beta(e) \in R_\mathbb{E}$ so that $1 + \beta(e) = \beta(x^{-1}S)$. However, $(1 + \alpha(S))(1 + \beta(e)) = \alpha(x)\beta(x^{-1}S) \in R_x R_{x^{-1}S} \subseteq R_S$. Thus, since $e \in S$, $(1 + \alpha(S))(1 + \beta(e)) = 1 + \gamma(S)$, where $\gamma(S) \in R_S$. Hence, for some $\delta(S) \in R_{x^{-1}S}$, $1 + \gamma(S) = \delta(S)$. In particular, $1 \in R_S$. Thus, since $S = \{e\}$ satisfies the hypothesis, $1 \in R_\mathbb{E}$.

Received by the editors October 26, 1984.

1980 Mathematics Subject Classification. Primary 16A03, 16A21.

©1985 American Mathematical Society

0002-9939/85 $1.00 + $.25 per page
Remark 1. Obviously the notion of G-system can be extended to infinite G. However, Theorem 1 does not hold in this case. Namely, let $A$ be the ring of all rational numbers of the form $2n/(2m + 1)$, where $n, m$ are integers and let $R = A[x]$ be the polynomial ring. There is a $Z$-gradation on $R$ such that $R_0 = A$. The homomorphism $f: R \rightarrow Q$ given by $f(w(x)) = w(\frac{1}{2})$ maps $R$ onto the field $Q$ of rational numbers. Thus $Q$ is a $Z$-system with unity 1 and $1 \notin A = Q_0$.

As a consequence of Theorem 1 we obtain

Corollary 1. Let $R = \sum_{g \in G} R_g$ be a G-system with unity 1. Then
(a) if $I$ is a right (left) ideal of $R_e$, then $IR = R$ ($RI = R$) if and only if $I = Re$,
(b) an element $x \in R_e$ is right (left) invertible in $R$ if and only if $x$ is right (left) invertible in $R_e$,
(c) $J(R) \cap R_e \subseteq J(R_e)$.

Proof. (a) If $I = Re$, then by Theorem 1, $1 \in I$ and $IR = R$. Conversely, let $R = \sum_{g \in G} IR_g$. Since, for all $g, h \in G$, $(IR_g)(IR_h) = I(R_g IR_h) \subseteq IR_{gh}$, $IR = \sum_{g \in G} IR_g$ is a G-system with unity. By Theorem 1, $1 \in IR_e \subseteq I$, so $I = Re$.

(b) Let $x \in R_e$ be right invertible in $R$. Consider a right ideal $I = xRe$ of $R_e$. Since $IR = R$, by (a) we have $I = Re$. Therefore, $xx' = 1$ for some $x' \in R_e$.

(c) By (b), $J(R) \cap R_e$ is a quasi-regular ideal of $R$, so $J(R) \cap R_e \subseteq J(R_e)$.

Remark 2. Corollary 1(c) holds also for G-systems without unity. Namely, let $R = \sum_{g \in G} R_g$ be any G-systems. Defining on the additive group $\hat{R} = R \times Z$ multiplication by $(x, m)(y, n) = (xy + nx + my, mn)$, we obtain a natural extension of $R$ to a ring with unity. Moreover, the ring $\hat{R}$ has a structure of G-system $\hat{R} = \sum_{g \in G} \hat{R}_g$, where $\hat{R}_e = R_e \times Z$ and, for $g \neq e$, $\hat{R}_g = R_g \times \{0\}$. It is clear that $J(R) = J(\hat{R})$ and $J(R_e) = J(\hat{R}_e)$. Thus, $J(R) \cap R_e = J(\hat{R}) \cap \hat{R}_e \subseteq J(\hat{R}_e) = J(R_e)$.

The following lemma is in fact the crucial step in the proof of a "Clifford type" theorem for G-systems.

Lemma 1. Let $M$ be a right module over G-system $R = \sum_{g \in G} R_g$ and let $0 \neq M_1 \subseteq M_2 \subseteq \cdots \subseteq M$ be a chain of $R_e$-submodules of $M_{R_e}$ such that $\bigcup_{n=1}^{\infty} M_n$ is essential in $M_{R_e}$. Then for some $m \geq 1$, $M_m$ contains a nonzero $R$-submodule of $M$.

Proof. Let us observe that using the procedure of Remark 2, we can reduce the proof to G-systems with unity. Now we shall prove by induction on $k = 1, 2, \ldots, |G|$ that there exist a subset $H_k \subseteq G$ and a nonzero element $m_k \in M$ such that
1°. $e \in H_k$,
2°. $|H_k| = k$,
3°. $0 \neq \sum_{h \in H_k} m_k R_h \subseteq M_{s(k)}$ for some $s(k) \geq 1$.

For $k = 1$ we put $H_1 = \{ e \}$, $m_1$ any nonzero element of $M_1$ and $s(1) = 1$. Let $|G| > k \geq 1$ and $m_k, H_k$ satisfy 1°–3°. Consider an element $g \notin H_k$. If $m_k R_g = 0$, then $m_{k+1} = m_k, H_{k+1} = H_k \cup \{ g \}$ satisfy 1°–3°. If $m_k R_g \neq 0$, then by essentiality of $\bigcup_{n=1}^{\infty} M_n$, there exists $r_g \in R_g$ such that $0 \neq m_k r_g \in M_t$ for some $t \geq 1$. Let
\[ m_{k+1} = m_k r^g, \quad H_{k+1} = g^{-1}H_k \cup \{e\} \text{ and } s(k + 1) = \max\{s(k), t\}. \] Clearly, \(|H_{k+1}| = k + 1\) and, since \(r^g R_{g^{-1}h} \subseteq R_g R_{g^{-1}h} \subseteq R_h\),

\[
0 \neq \sum_{h \in H_{k+1}} m_{k+1}R_h = m_{k+1}R_e + \sum_{h \in H_k} m_k r^g R_g - 1_h
\subseteq M_t + \sum_{h \in H_k} m_k R_h \subseteq M_t + M_{s(k)} = M_{s(k+1)}.
\]

Applying 3° to \(k = |G|\) we obtain that, for some \(m \geq 1\), \(M_m\) contains a nonzero \(R\)-submodule of \(M_R\).

Now we can prove a “Clifford type” theorem for \(G\)-systems.

**Theorem 2.** If \(M\) is a simple right \(R\)-module, then \(M_R = M_1 \oplus \cdots \oplus M_k\) is a direct sum of \(k \leq |G|\) simple \(R_e\)-modules.

**Proof.** Let us observe that any nonzero \(R_e\)-submodule \(N\) of \(M\) is its direct summand. Indeed, let \(K\) be a maximal with respect to \(N \cap K = 0\) submodule of \(M_R\). Then \(N \oplus K\) is an essential submodule of \(M_R\). Since \(M\) is a simple \(R\)-module, by Lemma 1 we obtain that \(N \oplus K = M\).

Thus every nonzero \(R_e\)-submodule of \(M\) contains a simple \(R_e\)-module. Let \(H\) be a subset of \(G\) containing \(e\), of maximal cardinality, such that, for some \(m \in M\),

\[\sum_{h \in H} mR_h \neq 0\]

and, for all \(h \in H\), \(mR_h = 0\) or \(mR_h\) is a simple \(R_e\)-module. We claim that \(H = G\). Indeed, if \(g \in G \setminus H\), then \(mR_g \neq 0\). Let \(\overline{mR_g}\) be a simple \(R_e\)-submodule of \(mR_g\). Then for \(h \in H\), \(\overline{mR_{g^{-1}h}} \subseteq mR_g R_{g^{-1}h} \subseteq mR_h\). Thus if \(\overline{h} \in g^{-1}H \cup \{e\}\), then \(\overline{mR_h} = 0\) or \(\overline{mR_h}\) is a simple \(R_e\)-module. This contradicts maximality of \(H\) and proves the claim.

Therefore, \(M\) is a sum of \(k \leq |G|\) simple \(R_e\)-modules.

Theorem 2 and Remark 2 imply immediately

**Corollary 2.** If \(R = \sum_{g \in G} R_g\) is a \(G\)-system, then

(a) for any right \(R\)-module \(M\), \(J(M_R) \subseteq J(M_R)\).

(b) for any right \(R\)-module \(M\), \(\text{Soc}(M_R) \subseteq \text{Soc}(M_R)\), where \(\text{Soc}(-)\) denotes the socle.

(c) (cf. [2]) \(J(R_e) = J(R) \cap R_e\).

The graded Jacobson radical \(J_G(R)\) of a \(G\)-graded ring \(\bigoplus_{g \in G} R_g\) is defined in [1] as the ideal of \(R\) satisfying the following equivalent conditions:

1. \(J_G(R)\) is the intersection of all maximal graded right ideals of \(R\).

2. \(J_G(R)\) is the largest graded ideal \(I\) of \(R\) such that \(I \cap R_e\) is a quasi-regular ideal of \(R_e\).

**Theorem 3.** (cf. [2, 6]). For every ring \(R\) graded by finite group \(G\), \(J_G(R) \subseteq J(R)\).

**Proof.** Obviously, \(I = RJ(R_e)\) is a graded ideal of \(R\). By Corollary 2(c), \(I \subseteq J(R)\) and \(J(R_e) \subseteq I \cap R_e \subseteq J(R) \cap R_e = J(R_e)\). Thus, \(I \cap R_e = J(R_e)\), so \(I \subseteq J_G(R)\). Consider the ring \(S = J_G(R)/I\). Clearly, \(S\) is a \(G\)-graded ring and its identity component \(S_e = 0\). Thus by [1], \(S^{|G|} = 0\). Therefore, \(J_G(R)\) is a \(J\)-radical ideal of \(R\) and \(J_G(R) \subseteq J(R)\).

We close this section by

**Theorem 4.** Let \(M\) be a right module over \(G\)-system \(R = \sum_{g \in G} R_g\). Then \(M_R\) is Noetherian if and only if \(M_{R_e}\) is Noetherian.
Proof. Suppose that there exists Noetherian $R$-module $M$ which is not Noetherian as $R_e$-module. Using Noetherian induction we may assume that, for each nonzero $R$-submodule $N$, the module $(M/N)_{R_e}$ is Noetherian.

Let $X_1 \subseteq X_2 \subseteq \cdots \subseteq M_{R_e}$ be a strictly ascending chain of $R_e$-submodules and let $Y$ be an $R_e$-submodule of $M$ maximal with respect to $(\bigcup_{n=1}^{\infty}X_n) \cap Y = 0$. Now we have a strictly ascending chain $X_1 \oplus Y \subseteq X_2 \oplus Y \subseteq \cdots$ of $R_e$-submodules of $M$ such that $\bigcup_{n=1}^{\infty}X_n \oplus Y = (\bigcup_{n=1}^{\infty}X_n) \oplus Y$ is essential in $M_{R_e}$. By Lemma 1, for some $m \geq 1$, the module $X_m \oplus Y$ contains a nonzero, say $N$, $R$-submodule. Therefore, in $(M/N)_{R_e}$, we have a strictly ascending chain $X_{m+1} \oplus Y/N \subseteq X_{m+2} \oplus Y/N \subseteq \cdots \subseteq M/N$, a contradiction.

The converse is clear.

Remark 3. Theorem 4 implies that if the ring $R$ is right Noetherian then the ring $R_e$ is right Noetherian. The converse is not true in general. Consider the matrix ring $\begin{pmatrix} \mathbb{K} & \mathbb{K} \end{pmatrix}$, where $\mathbb{K}$ is a field and $\mathbb{A}$ an arbitrary infinite-dimensional $\mathbb{K}$-algebra. Putting $R_0 = \begin{pmatrix} \mathbb{K} & 0 \\ 0 & \mathbb{K} \end{pmatrix}$, $R_1 = \begin{pmatrix} 0 & \mathbb{A} \\ 0 & 0 \end{pmatrix}$, we obtain a $C_2$-gradation on $R$. Clearly, $R_0$ is right Noetherian but $R$ contains a strictly ascending chain of right ideals of $R$, e.g. $\begin{pmatrix} 0 & V_1 \\ 0 & 0 \end{pmatrix} \subseteq \begin{pmatrix} 0 & V_2 \\ 0 & 0 \end{pmatrix} \subseteq \cdots$, where $V_1 \subseteq V_2 \subseteq \cdots \subseteq \mathbb{A}$ is a chain of $\mathbb{K}$-subspaces of $\mathbb{A}$.

2. In this section we show that some of the previous results can be applied to bimodules. Moreover, we obtain a link between both Brown-McCoy radicals and prime ideals of $R_e$ and $R$. Now, G-systems do not necessarily have a unity element.

Let us observe that the ring $S = R^0 \otimes_{\mathbb{Z}} R$ ($R^0$ is the opposite ring of $R$) has a structure of the $G \times G$-system. For $((g, h)) \in G \times G$ we make $S_{(g, h)}$ the additive subgroup of $S$ generated by $r_g^0 \otimes r_h$, where $r_g^0 \in R^0_g$, $r_h \in R_h$. Clearly, an $R$-subbimodule, or an $R_e$-subbimodule of an $R$-bimodule $V$, is simply a right submodule over $S$ or $S_{(e, e)}$, respectively. Thus, by Theorem 2, we have

Corollary 3. If $V$ is a simple $R$-bimodule, then $R_e V_{R_e} = V_1 \oplus \cdots \oplus V_k$ is a direct sum of $k \leq |G|^2$ simple $R_e$-bimodules.

Corollary 4. If $J$ is a maximal ideal of the G-system $R = \sum_{g \in G} R_g$, then $J \cap R_e$ is a finite intersection of $k \leq |G|^2$ maximal ideals of $R_e$.

Proof. Consider the G-system $\overline{R} = R/J$. Obviously, $\overline{R}$ is a simple $R$-bimodule. By Corollary 3, there exist simple $R_e$-subbimodules $\overline{V}_1, \ldots, \overline{V}_k$ ($k \leq |G|^2$) such that $R_e \overline{R} = \overline{V}_1 \oplus \cdots \oplus \overline{V}_k$. Thus, for $\overline{W}_i = \overline{V}_1 \oplus \cdots \oplus \overline{V}_{i-1} \oplus \overline{V}_{i+1} \oplus \cdots \oplus \overline{V}_k$, $0 = \bigcap_{i=1}^{k} \overline{W}_i$ and $\overline{W}_i$ are maximal $R_e$-subbimodules of $R_e \overline{R}$. Therefore, the ideal $J$ is an intersection of maximal $R_e$-subbimodules $W_1, \ldots, W_k$ of $R$ and $W_i \cap R_e$ ($i = 1, 2, \ldots, k$) are maximal ideals of $R_e$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Let, for a ring $R$, $U(R)$ denote the Brown-McCoy radical of $R$, i.e. the intersection of all ideals $I$ of $R$ such that $R/I$ is a simple ring with unity.

**Theorem 5.** For every $G$-system $R = \sum_{g \in G} R_g$, $U(R_e) \subseteq U(R)$.

**Proof.** Let $I$ be a maximal ideal of $R$ such that $\overline{R} = R/I$ is a simple ring with unity. Since $\overline{R}$ is a $G$-system, by Theorem 1 the ring $\overline{R} = R_e/I \cap R_e$ has unity. By Corollary 4, there exist maximal ideals $I_1, \ldots, I_k$ of $R_e$ such that $I \cap R_e = I_1 \cap \cdots \cap I_k$. Obviously, for $i = 1, \ldots, k$, $R_e/I_i$ are simple rings with unity, so $U(R_e) \subseteq U(R)$.

We now give an example of a $C_2$-graded ring $R = R_0 \oplus R_1$ with $U(R_e) \neq U(R) \cap R_e$.

**Example.** Let $V$ be an infinite-dimensional vector space over a field $k$ and let $V_0, V_1$ be subspaces of $k$ such that $\dim_k V_0 = 1$ and $V_0 \oplus V_1 = V$. Consider the ring $R$ of all linear transformations of finite rank of $V$. $R$ is a simple ring without unity, so $U(R) = R$. On $R$ we define a $C_2$-gradation putting

$$
R_0 = \{ f \in R \mid f(V_0) \subseteq V_0, f(V_1) \subseteq V_1 \},
$$
$$
R_1 = \{ f \in R \mid f(V_0) \subseteq V_1, f(V_1) \subseteq V_0 \}.
$$

Let us observe that $R_0$ is isomorphic to the ring $k \oplus R$. Hence $R_0$ is not a $U$-radical ring.

In [5] Passman proved that if $R$ is a prime ring then there exist sets $Y, Z$ such that the power series-polynomial ring $R^* = (R\langle\langle Y\rangle\rangle)\langle\langle Z\rangle\rangle$ is primitive. Clearly, if $R = \sum_{g \in G} R_g$ is a $G$-system, then $R^* = \sum_{g \in G} R_g^*$ is a $G$-system, where $R_g^*$ denotes the set of all power series-polynomials with coefficients in $R_g$. Using Passman's method (see [5, Theorem 3.1]) and Theorem 2 we obtain

**Theorem 6 (cf. [2]).** If $P$ is a prime ideal of $R$, then $P \cap R_e$ is a finite intersection of $k \leq |G|$ prime ideals of $R_e$.

As an immediate consequence of Theorem 6 we have

**Corollary 5 (cf. [2]).** $B(R_e) = B(R) \cap R_e$, where $B(-)$ denotes prime radical.

**Acknowledgment.** I am deeply grateful to Dr. E. R. Puczyłowski for his valuable suggestions and for stimulating discussions. I would also like to thank the editor for suggestions making the proof of Theorem 1 clearer.

**References**


**Institute of Mathematics, University of Warsaw, Białystok Division, Akademicka 2, 15–267 Białystok, Poland**