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COINCIDENCE SETS IN THE OBSTACLE PROBLEM FOR THE p-HARMONIC OPERATOR

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ABSTRACT. We consider the obstacle problem for the p-harmonic operator

$$\operatorname{div}\left(|\nabla \cdot|^{p-2}\nabla \cdot\right) \quad \text{with } p > 1,$$

and show that the coincidence set is star shaped under certain conditions on the obstacle.

1. Introduction. In the previous paper [**13**], we considered the obstacle problem for the harmonic operator which H. Lewy and G. Stampacchia treated in [**10**], and using an idea of L. Caffarelli and J. Spruck [**2**] we showed that the coincidence set is star shaped under certain conditions on the obstacle. The purpose of this paper is to show that the same is true also in the case of the p-harmonic operator with $p > 1$.

Recently the obstacle problem for the p-harmonic operator was considered by C. Bandle and J. Mossino [**1**], S. Granlund, P. Lindqvist and O. Martio [**5**], and P. Lindqvist [**11**]. In the case of the harmonic operator the regularity for the obstacle problem is well known, but in the case of the p-harmonic operator the regularity is not known.

On the other hand, the $C^{1+\alpha}$-estimates of solutions to the p-harmonic equation with $p > 1$,

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = f,$$

were obtained recently by P. Tolksdorf [**14, 15**], J. Lewis [**9**], and E. DiBenedetto [**3**].

We use the estimates of P. Tolksdorf [**15**], and obtain the $C^{1+\alpha}$-regularity for the obstacle problem with a concave obstacle. Furthermore, using an idea of J. Lewis [**8**], we show that the solution to our obstacle problem is real analytic in the noncoincidence set. Proceeding as in the case of the harmonic operator, we obtain the starshapedness of the coincidence set.

Finally we note that starshapedness of level sets of the solution to the obstacle problem with $p = 2$ was proved by B. Kawohl [**6**].

2. Result. Let Ω be a bounded convex domain in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$. Suppose that the origin 0 is contained in Ω. Let $f \in C^1(\bar{\Omega}) \cap C^2(\bar{\Omega} - \{0\})$ be a nonnegative convex function which is positive on $\partial\Omega$ and homogeneous of degree $s > 1$. Consider the obstacle $\psi \in C^1(\bar{\Omega}) \cap C^2(\bar{\Omega} - \{0\})$, which is negative on $\partial\Omega$,

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defined by

$$\psi(x) = -f(x) + c \tag{2.1}$$

with positive constant c. We fix a number $p > 1$ and consider the following obstacle problem.

$$\begin{aligned}&\text{Find a function } u \text{ in the closed convex set}\\ &\mathbf{K}(\psi) = \{u \in W_0^{1,p}(\Omega);\ u \geqslant \psi \text{ in } \Omega\}\end{aligned} \tag{2.2}$$

which minimizes the integral $\int_\Omega |\nabla u|^p\, dx$.

It follows from the direct method in the calculus of variations that there exists a solution to this problem (see J. Lewis [**8**] for proof). We can prove the uniqueness by using the variational inequality

$$\begin{aligned}&\text{Find } u \in \mathbf{K}(\psi) \text{ such that}\\ &\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla(v - u)\, dx \geqslant 0 \quad \text{for all } v \in \mathbf{K}(\psi),\end{aligned} \tag{2.3}$$

which is equivalent to (2.2). Let u be the solution to (2.2). Denote by $I(\psi)$ its coincidence set, namely,

$$I(\psi) = \{x \in \Omega : u(x) = \psi(x)\}. \tag{2.4}$$

Now our result is

THEOREM. *The coincidence set $I(\psi)$ is starshaped with respect to the origin.*

3. Preliminaries. We quote a lemma from [**14**] (see [**14**, p. 129, Lemma 1] for proof).

LEMMA 3.1. *For $q, \xi \in \mathbf{R}^n$ we have*

$$\left(|q|^{p-2} q - |\xi|^{p-2}\xi\right) \cdot (q - \xi) \geqslant \begin{cases} \frac{1}{4}(p-1)(1 + |q| + |\xi|)^{p-2}|q - \xi|^2 & \text{if } p \leqslant 2, \\ \left(\frac{1}{4}\right)^{p-1}|q - \xi|^p & \text{if } p \geqslant 2. \end{cases}$$

The uniqueness of the solution to (2.2) follows from this lemma and the variational inequality (2.3) directly.

With the help of Lemma 3.1 we obtain a characterization of the solution u to (2.2) by the same argument as in [**7**, p. 41, Theorem 6.4].

PROPOSITION 3.2. *Let $g \in W^{1,p}(\Omega)$ be a super p-harmonic function in Ω (that is, $\operatorname{div}(|\nabla g|^{p-2}\nabla g) \leqslant 0$ in the weak sense) satisfying $g \geqslant \psi$ in Ω and $g \geqslant 0$ on $\partial\Omega$. Then we have $u \leqslant g$ in Ω.*

4. Regularity of the solution to (2.2). First of all, we prove

PROPOSITION 4.1. *There exists a number $r > 0$ such that $B_r(0) \subset I(\psi)$ where $B_r(0)$ denotes an open ball in $\mathbf{R}^n$ centered at the origin with radius r.*

PROOF. We prove this by the same argument as in [**7**, p. 176]. Let I_1 be the set of points $y \in \Omega$ for which the tangent plane of the graph $(\cdot, \psi(\cdot))$ at $(y, \psi(y))$,

$$\Pi_y : x_{n+1} = W_y(x) = \nabla\psi(y) \cdot (x - y) + \psi(y),$$

does not meet $\Omega \times \{0\}$. Since $\max_\Omega \psi(x) = \psi(0) = c > 0$, we see that $0 \in I_1$. Furthermore, since $\psi \in C^1(\bar{\Omega})$, so I_1 contains an open neighborhood of 0. Let $y \in I_1$. Since ψ is concave, we see that $W_y \geqslant \psi$ in Ω. Of course, $W_y \geqslant 0$ on $\partial\Omega$ and W_y is p-harmonic in Ω. Therefore, by Proposition 3.2, we get $u \leqslant W_y$ in Ω. Since $\psi(y) \leqslant u(y) \leqslant W_y(y) = \psi(y)$, so $y \in I(\psi)$. Then $I_1 \subset I(\psi)$. This completes the proof.

Using the penalty method we obtain

PROPOSITION 4.2. *The solution u belongs to $C^{1+\alpha}(\bar{\Omega} - \{0\})$ for some $0 < \alpha < 1$.*

PROOF. In view of Proposition 4.1 it suffices to show that the solution u belongs to $C^{1+\alpha}(\bar{\Omega} - B_{r/2}(0))$. Let $B = B_{r/4}(0)$. As in [7, p. 108] we specify, for $\varepsilon > 0$,

$$\vartheta_\varepsilon(t) = \begin{cases} 1 & \text{if } t \leqslant 0, \\ 1 - (t/\varepsilon) & \text{if } 0 < t \leqslant \varepsilon, \\ 0 & \text{if } t > \varepsilon, \end{cases}$$

and consider the penalized problem with $0 < \varepsilon < 1$.

$$\begin{aligned} &\text{Find } u_\varepsilon \in W^{1,p}(\Omega - B) \text{ satisfying} \\ &\operatorname{div}\left(|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\right) = \operatorname{div}\left(|\nabla\psi|^{p-2}\nabla\psi\right)\vartheta_\varepsilon(u_\varepsilon - \psi) \quad \text{in } \Omega - B, \\ &u_\varepsilon = 0 \quad \text{on } \partial\Omega \quad \text{and} \quad u_\varepsilon = \psi \quad \text{on } \partial B. \end{aligned} \tag{4.1}$$

The existence of u_ε is due to the variational method. Precisely, we consider the functional

$$\begin{aligned} J(v) = (1/p)\int_{\Omega - B} |\nabla v|^p \, dx \\ + \int_{\Omega - B} \left\{ \operatorname{div}\left(|\nabla\psi|^{p-2}\nabla\psi\right) \int_\varepsilon^{v-\psi} \vartheta_\varepsilon(t)\, dt \right\} dx \end{aligned}$$

over the class of admissible functions

$$G = \left\{ v \in W^{1,p}(\Omega - B);\ v = 0 \text{ on } \partial\Omega \text{ and } v = \psi \text{ on } \partial B \right\}.$$

In view of the definition of ψ in (2.1) we see that

$$-C \leqslant \operatorname{div}\left(|\nabla\psi|^{p-2}\nabla\psi\right) \leqslant -\mu < 0 \quad \text{in } \Omega - B \tag{4.2}$$

with some positive constants C and μ. Then it follows that $J(v)$ is nonnegative on G and the functional J is lower semicontinuous. Hence there exists a function $w \in G$ satisfying $J(w) = \min_{v \in G} J(v)$ (see [**8**, p. 203] for proof), and this function w is also a solution to (4.1). By virtue of (4.2) and the fact that ϑ_ε is decreasing, using the weak comparison principle due to P. Tolksdorf (see [**15**, pp. 800–801, Lemma 3.1]), we obtain the uniqueness to the solution to (4.1). Thus there exists a unique solution to (4.1), say u_ε.

Here, we see that this solution u_ε has the following properties:

(4.3) $0 \leqslant u_\varepsilon \leqslant \max_\Omega \psi = c$ in $\Omega - B$,

(4.4) there exist positive constants C_0 and α independent of ε, which satisfy $\|u_\varepsilon\|_{C^{1+\alpha}(\bar{\Omega} - B_{r/2}(0))} \leqslant C_0$, and

(4.5) $u \leqslant u_\varepsilon \leqslant u + \varepsilon$ in $\Omega - B$.

It follows from the comparison principle due to P. Tolksdorf that the inequality (4.3) holds. In view of (4.1), (4.2), and (4.3), we get (4.4) by using the $C^{1+\alpha}$-estimate due to P. Tolksdorf (see [**15**, p. 806, Proposition 3.7]). With the help of Lemma 3.1, we get the inequality (4.5) by the same argument as in [**7**, pp. 109–111].

Now, since the imbedding $C^{1+\alpha} \hookrightarrow C^1$ is compact, the bounded family $\{u_\varepsilon\}$ in $C^{1+\alpha}(\overline{\Omega} - B_{r/2}(0))$ has a subsequence $\{u_{\varepsilon'}\}$ which converges to a function $u^{\sim}$ in $C^1(\overline{\Omega} - B_{r/2}(0))$. By virtue of (4.5), we get $u^{\sim} = u$. It follows from (4.4) that u belongs to $C^{1+\alpha}(\overline{\Omega} - B_{r/2}(0))$. This completes the proof.

In particular, from Proposition 4.2 we see that $I(\psi)$ is a closed set. Furthermore we show

PROPOSITION 4.3. $\inf_{\Omega - I(\psi)} |\nabla u| > 0$.

PROOF. Recalling an idea of J. Lewis [**8**], we show that

$$x \cdot \nabla u(x) < 0 \quad \text{in } \overline{\Omega - I(\psi)}\,.$$

Since the C^1-function $u - \psi$ attains its minimum at any point in $I(\psi)$, it follows that $x \cdot \nabla u(x) = x \cdot \nabla \psi(x)$ in $I(\psi)$. In view of the definition of ψ, we obtain $x \cdot \nabla \psi(x) = -sf(x) < 0$ in $\Omega - \{0\}$. Then we have

$$x \cdot \nabla u(x) < 0 \quad \text{on } \partial I(\psi).$$

Therefore, since Ω is convex and $u > 0$ is p-harmonic in $\Omega - I(\psi)$, proceeding as in [**8**, pp. 207–208], we get $x \cdot \nabla u(x) < 0$ on $\partial\Omega$ and conclude that $x \cdot \nabla u(x) < 0$ in $\overline{\Omega - I(\psi)}$.

By Propositions 4.2 and 4.3, and the regularity theory for the elliptic partial differential equation, we see that the solution u is real analytic in $\Omega - I(\psi)$ (see [**4** and **12**]).

5. Proof of Theorem. First of all, as in [**13**] we introduce the function v,

$$v(x) = x \cdot \nabla(u - \psi)(x) - s(u - \psi)(x) = x \cdot \nabla u(x) - su(x) + sc. \tag{5.1}$$

Then v is continuous in $\overline{\Omega}$ by Proposition 4.2. Here it suffices to show that v is nonnegative in $\Omega - I(\psi)$. Indeed, suppose that this is true. Then, since $u - \psi > 0$ in $\Omega - I(\psi)$, we have

$$x \cdot \nabla(u - \psi)(x) > 0 \quad \text{in } \Omega - I(\psi).$$

Therefore, in view of the definition of $I(\psi)$ (see (2.4)), we see that the coincidence set $I(\psi)$ is starshaped with respect to the origin.

We observe that

$$v = 0 \quad \text{in } I(\psi). \tag{5.2}$$

With the help of Proposition 3.2 we obtain by the same argument as in [**13**, Lemma 2.1]

$$v > 0 \quad \text{on } \partial\Omega. \tag{5.3}$$

By virtue of the regularity of u and Proposition 4.3, we obtain

$$|\nabla u|^2 \Delta u + (p-2)\sum D_i u D_j u D_{ij} u = 0 \quad \text{in } \Omega - I(\psi),$$

since u is p-harmonic in $\Omega - I(\psi)$. Here $D_i = \partial/\partial x_i$ and $D_{ij} = \partial^2/\partial x_i \partial x_j$. Applying the differential operator $x \cdot \nabla$ to this and using (5.1), we obtain

$$Lv = \sum a^{ij}(x) D_{ij} v + \sum b^j(x) D_j v = 0 \quad \text{in } \Omega - I(\psi), \tag{5.4}$$

where $a^{ij}(x) = |\nabla u(x)|^2 \delta_{ij} + (p-2) D_i u(x) D_j u(x)$ (δ_{ij} is Kronecker's symbol) and

$$b^j(x) = 2\left\{ D_j u(x) \Delta u(x) + (p-2) \sum D_k u(x) D_{jk} u(x) \right\}.$$

Observing that the operator L is a uniformly elliptic operator with locally bounded coefficients in $\Omega - I(\psi)$, we have from (5.2), (5.3), (5.4), and the maximum principle $v \geqslant 0$ in $\Omega - I(\psi)$. This completes the proof.

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