WEAK SPECTRAL THEORY

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Abstract. We initiate the weak spectrum of a linear operator on $L^p$ spaces, $1 < p < \infty$. The weak spectrum of a pseudo-differential operator with symbol in $S^m_{\rho,0}$, where $-\infty < \rho < \infty$ and $0 < \rho < 1$, is investigated.

1. Introduction. For $m \in (-\infty, \infty)$ and $\rho \in [0, 1]$, we define $S^m_{\rho,0}$ to be the set of all functions $\sigma$ in $C^\infty(R^n)$ such that, for each multi-index $\alpha$, $(D^\alpha \sigma)(\xi) = O(|\xi|^{m-\rho|\alpha|})$ as $|\xi| \to \infty$. Let $\sigma \in S^m_{\rho,0}$. Then we define the pseudo-differential operator $T_\sigma$, initially on $\mathcal{S}$ (the Schwartz space), by

$$(T_\sigma \varphi)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \sigma(\xi) \hat{\varphi}(\xi) \, d\xi$$

for all $\varphi \in \mathcal{S}$. Here, $\hat{\cdot}$ denotes the Fourier transformation. Obviously $T_\sigma$ maps $\mathcal{S}$ into $\mathcal{S}$. It can be shown (see Proposition 2.1 in Wong [5]) that, for $1 \leq p \leq \infty$, $T_\sigma : \mathcal{S} \to \mathcal{S}$ is closable in $L^p(R^n)$. We denote the closure by $T_{\sigma p}$. Detailed information about the spectrum $\Sigma(T_{\sigma p})$ of $T_{\sigma p}$ can be found in Wong [3, 4, 5]. The corresponding results for partial differential operators have been gathered in Schechter [2].

2. The weak spectrum. Let $A$ be a closed linear operator defined on $L^p(R^n)$. Then a complex number $\lambda$ is said to be in the weak resolvent set $\rho_w(A)$ of $A$ if the range $R(A - \lambda)$ of $A - \lambda$ is dense in $L^p(R^n)$ and there is a constant $C > 0$ such that

$$m \{ x \in R^n : |\varphi(x)| > \alpha \} \leq \left\{ C \| (A - \lambda) \varphi \| / \alpha \right\}^p$$

for all $\varphi > 0$ and $\varphi$ in the domain $\mathcal{D}(A)$ of $A$. Here, $m \{ \cdots \}$ denotes the Lebesgue measure of $\{ \cdots \}$ and $\| \| \text{ the } L^p$ norm. As usual, the weak spectrum $\Sigma_w(A)$ of $A$ is defined to be $C - \rho_w(A)$. Obviously, $\Sigma_w(A) \subseteq \Sigma(A)$. That $\Sigma_w(A)$ can be a proper subset of $\Sigma(A)$ will be shown in §5.

3. On $\Sigma_w(T_{\sigma p})$, $1 \leq p \leq \infty$. We first show that $\Sigma_w(T_{\sigma p})$ is not empty.

Theorem 3.1. If $\sigma(\xi)$ is not bounded away from a complex number $\lambda$ for all $\xi \in \mathbb{R}^n$, then $\lambda \in \Sigma_w(T_{\sigma p})$.

Proof. For simplicity, we suppose that $\lambda = 0$. Let $\{ \xi_k \}$ be a sequence of elements of $\mathbb{R}^n$ such that $\sigma(\xi_k) \to 0$ as $k \to \infty$. Let $\{ \varepsilon_k \}$ be a sequence of positive numbers. Let $\theta \in C_0^\infty(R^n)$ be such that $\theta(\xi) = 0$ for $|\xi| > 1$ and $(2\pi)^{-n/2} \int_{\mathbb{R}^n} \theta(\xi) \, d\xi = 1$. Let

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ψ ∈ ℘ be such that ˆψ = θ. For k = 1, 2, ..., define

$$\varphi_k(x) = e^{\pi i/p^k} \psi(e_k x)e^{i\xi_k \cdot x}.$$  

If 0 ∈ ρ(\(T_{op}\)), then there is a constant C > 0 such that

$$m \left\{ x \in \mathbb{R}^n : |\varphi_k(x)| > \alpha \right\} \leq \left\{ C \| T_{op} \varphi_k \| / \alpha \right\}^p$$

for all α > 0 and k = 1, 2, .... Choosing α = \(\frac{1}{2} \epsilon_k^{n/p}\) and using (3.1), inequality (3.2) becomes

$$m \left\{ x \in \mathbb{R}^n : |\varphi_k(x)| > \frac{1}{2} \right\} \leq \epsilon_k^{n/p} O \left( \| T_{op} \varphi_k \|^p \right).$$

Since \(\psi(0) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \theta(\xi) \, d\xi = 1\), it follows that there exists a δ > 0 such that

$$|\varphi(x)| > \frac{1}{2} \text{ whenever } |x| < \delta.$$  

Therefore

$$m \left\{ x \in \mathbb{R}^n : |\varphi(e_k x)| > \frac{1}{2} \right\} \geq \pi(\delta/\epsilon_k)^n$$

for k = 1, 2, .... Hence, by (3.3),

$$\pi \delta^n \leq O \left( \| T_{op} \varphi_k \|^p \right)$$

for k = 1, 2, .... But as has been shown in the proof of Theorem 3.1 in Wong [5], we can choose the \(\epsilon_k\)'s going to zero so fast that \(\| T_{op} \varphi_k \| \to 0\) as \(k \to \infty\). Thus (3.4) is impossible.

A useful consequence of Theorem 3.1 is

**Corollary 3.2.** \(\Sigma_w(T_{op})\) contains the set \(\{ \sigma(\xi) : \xi \in \mathbb{R}^n \}\).

4. **Multipliers of weak type** (p, p), 1 ≤ p < ∞. Let m be a bounded measurable function on \(\mathbb{R}^n\). For any \(\varphi \in \mathcal{S}\), we define \(T_m \varphi\) by

$$(T_m \varphi)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix \cdot \xi} m(\xi) \hat{\varphi}(\xi) \, d\xi.$$  

Suppose that there is a constant C > 0 such that

$$m \left\{ x \in \mathbb{R}^n : |(T_m \varphi)(x)| > \alpha \right\} \leq \left\{ C \| \varphi \| / \alpha \right\}^p$$

for all α > 0 and \(\varphi \in \mathcal{S}\). Then we call \(T_m\) a multiplier of weak type (p, p).

The connection between weak type multipliers and weak spectra is provided by

**Theorem 4.1.** A complex number λ is in the weak resolvent set ρ(\(T_{op}\)) of \(T_{op}\) if and only if \(1/(\sigma(\xi) - \lambda)\) is a multiplier of weak type (p, p).

**Proof.** We first prove necessity. Again for simplicity, let λ = 0. By Theorem 3.1, \(\sigma(\xi)\) is bounded away from 0 for all \(\xi \in \mathbb{R}^n\). For any \(f \in \mathcal{S}\), define u by \(\hat{u}(\xi) = \hat{f}(\xi)/\sigma(\xi)\). Then \(u \in \mathcal{S}\) and \(T_0 u = f\). Since 0 ∈ ρ(\(T_{op}\)), it follows that there is a constant C > 0 such that

$$m \left\{ x \in \mathbb{R}^n : |u(x)| > \alpha \right\} \leq \left\{ C \| f \| / \alpha \right\}^p$$

for all α > 0 and \(f \in \mathcal{S}\). Hence \(1/\sigma(\xi)\) is a multiplier of weak type (p, p). Conversely, if \(1/\sigma(\xi)\) is a multiplier of weak type (p, p), then there is a constant C > 0 such that

$$m \left\{ x \in \mathbb{R}^n : |\varphi(x)| > \alpha \right\} \leq \left\{ C \| T_0 \varphi \| / \alpha \right\}^p$$
for all $\alpha > 0$ and $\varphi \in \mathcal{S}$. Since $\sigma(\xi)$ is bounded away from 0 for all $\xi \in \mathbb{R}^n$, it follows that $\mathcal{S} \subseteq R(T_{\alpha p})$. This proves that $R(T_{\alpha p})$ is dense in $L^p(\mathbb{R}^n)$. Consequently, $0 \in \rho_w(T_{\alpha p})$ if we can show that (4.1) is valid for all $\varphi \in \mathcal{D}(T_{\alpha p})$.

**Lemma 4.2.** Inequality (4.1) is valid for all $\varphi \in \mathcal{D}(T_{\alpha p})$.

**Proof.** For any $\varphi \in \mathcal{D}(T_{\alpha p})$, let $\{\varphi_k\}$ be a sequence of functions in $C_0^\infty(\mathbb{R}^n)$ such that $\varphi_k \to \varphi$ and $T_{\alpha p}\varphi_k \to T_{\alpha p}\varphi$ in $L^p(\mathbb{R}^n)$ as $k \to \infty$. Pick a subsequence of $\{\varphi_k\}$, again denoted by $\{\varphi_k\}$, such that $\varphi_k \to \varphi$ a.e. on $\mathbb{R}^n$. For any $\alpha > 0$, we set

$$E(\alpha) = \{x \in \mathbb{R}^n: |\varphi(x)| > \alpha\}$$

and

$$E_k(\alpha) = \{x \in \mathbb{R}^n: |\varphi_k(x)| > \alpha\}$$

for $k = 1, 2, \ldots$. Since $\varphi \in L^p(\mathbb{R}^n)$, it follows that $m(E(\alpha)) < \infty$. So for any $\varepsilon > 0$, by Egoroff's Theorem, we can find a measurable set $A_\varepsilon \subseteq \mathbb{R}^n$ such that $m(A_\varepsilon) < \varepsilon$ and $\varphi_k(x) \to \varphi(x)$ uniformly for all $x \in E(\alpha) - A_\varepsilon$. Hence, there exists a positive integer $K$ such that $|\varphi_k(x)| > \alpha$ whenever $k \geq K$. For such $k$'s, $E(\alpha) - A_\varepsilon \subseteq E_k(\alpha)$, and using (4.1) and letting $k \to \infty$, we get

$$m(E(\alpha)) - \varepsilon \leq \left\{ C \|T_{\alpha p}\varphi\|/\alpha \right\}^p$$

for every $\alpha > 0$. Since $\varepsilon$ is an arbitrary positive number, the proof is complete.

5. **An example.** We begin with an observation.

**Lemma 5.1.** For any $p$ such that $1 \leq p < \infty$, a sufficient condition for $\Sigma_w(T_{\alpha p}) = \Sigma(T_{\alpha p})$ is that $\Sigma(T_{\alpha p}) = \{\sigma(\xi): \xi \in \mathbb{R}^n\}$.

Lemma 5.1 follows immediately from Corollary 3.2 and the fact that $\Sigma_w(T_{\alpha p}) \subseteq \Sigma(T_{\alpha p})$.

Let $\tau$ be the function defined by $\tau(\xi) = e^{i\xi \cdot \tau}/(1 + |\xi|^c)$, where $0 < a < 1$ and $0 < c < na/2$. Then, defining $\sigma$ by $\sigma(\xi) = 1/\tau(\xi)$, it is clear that $\sigma \in S_{-a,0}$. As has been proved in Wong [3, 4], $\Sigma(T_{\alpha p}) = \{\sigma(\xi): \xi \in \mathbb{R}^n\}$ if $p$ is any number such that $1 < p < \infty$ and $|1/p - 1/2| < c/na$. Thus, $\Sigma(T_{\alpha p}) = C$ if $|1/p - 1/2| > c/na$. The following result tells us that we know exactly what $\Sigma_w(T_{\alpha p})$ is if $1 < p < \infty$.

**Theorem 5.2.** $\Sigma_w(T_{\alpha p}) = \Sigma(T_{\alpha p})$, $1 < p < \infty$.

**Proof.** In view of Lemma 5.1, we need only consider $|1/p - 1/2| > c/na$. We first suppose that $1/p > 1/2 + c/na$. If $\lambda \in \Sigma(T_{\alpha p})$ and $\lambda \not\in \Sigma_w(T_{\alpha p})$, then by Theorem 4.1, $1/(\sigma(\xi) - \lambda)$ is a multiplier of weak types $(2,2)$ and $(p,p)$. Hence, by the Marcinkiewicz interpolation theorem, $1/(\sigma(\xi) - \lambda)$ is an $L^q$ multiplier for any $q$ such that $1/2 + c/na < 1/q < 1/p$. Thus, by Theorem 3.3 in Wong [3], $\lambda \in \rho(T_{\alpha q})$. But Theorem 3.1 in Wong [4] says that the spectrum of $T_{\alpha q}$ is either $C$ or $\{\sigma(\xi): \xi \in \mathbb{R}^n\}$. Hence, $\Sigma(T_{\alpha q}) = \{\sigma(\xi): \xi \in \mathbb{R}^n\}$. This is a contradiction. The proof for the case when $1/p < 1/2 - c/na$ is similar.

**Theorem 5.3.** $\Sigma_w(T_{\alpha 1}) = \{\sigma(\xi): \xi \in \mathbb{R}^n\}$. 
Proof. By Theorems 3.1 and 4.1, we need only show that if \( \lambda \in \mathbb{C} \) is such that \( \sigma(\xi) \neq \lambda \) for all \( \xi \in \mathbb{R}^n \), \( 1/(\sigma(\xi) - \lambda) \) is a multiplier of weak type \((1,1)\). But an easy computation shows that \( 1/(\sigma(\xi) - \lambda) \in S_{1-\alpha,0}^\sigma \), and hence it follows from Theorem 2' in Fefferman [1] that \( 1/(\sigma(\xi) - \lambda) \) is indeed a multiplier of weak type \((1,1)\).

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References


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