WEAK SPECTRAL THEORY

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ABSTRACT. We initiate the weak spectrum of a linear operator on $L^p$ spaces, $1 < p < \infty$. The weak spectrum of a pseudo-differential operator with symbol in $S^m_{\rho,0}$, where $-\infty < m < \infty$ and $0 < \rho < 1$, is investigated.

1. Introduction. For $m \in (-\infty, \infty)$ and $\rho \in [0,1]$, we define $S^m_{\rho,0}$ to be the set of all functions $\sigma$ in $C^\infty(\mathbb{R}^n)$ such that, for each multi-index $\alpha$, $(D^\alpha \sigma)(\xi) = O(|\xi|^{m-\rho|\alpha|})$ as $|\xi| \to \infty$. Let $\sigma \in S^m_{\rho,0}$. Then we define the pseudo-differential operator $T_{\sigma}$, initially on $\mathcal{S}$ (the Schwartz space), by

$$
(T_{\sigma} \varphi)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i \xi \cdot x} \sigma(\xi) \hat{\varphi}(\xi) \, d\xi
$$

for all $\varphi \in \mathcal{S}$. Here, $\hat{}$ denotes the Fourier transformation. Obviously $T_{\sigma}$ maps $\mathcal{S}$ into $\mathcal{S}$. It can be shown (see Proposition 2.1 in Wong [5]) that, for $1 \leq p \leq \infty$, $T_{\sigma}$ is closable in $L^p(\mathbb{R}^n)$. We denote the closure by $T_{\sigma,c}$. Detailed information about the spectrum $\Sigma(T_{\sigma,c})$ of $T_{\sigma,c}$ can be found in Wong [3, 4, 5]. The corresponding results for partial differential operators have been gathered in Schechter [2].

2. The weak spectrum. Let $A$ be a closed linear operator defined on $L^p(\mathbb{R}^n)$. Then a complex number $\lambda$ is said to be in the weak resolvent set $\rho_w(A)$ of $A$ if the range $R(A - \lambda)$ of $A - \lambda$ is dense in $L^p(\mathbb{R}^n)$ and there is a constant $C > 0$ such that

$$
m \{ x \in \mathbb{R}^n : |\varphi(x)| > \alpha \} \leq \left\{ C \| (A - \lambda) \varphi \| / \alpha \right\}^p
$$

for all $\alpha > 0$ and $\varphi$ in the domain $\mathcal{D}(A)$ of $A$. Here, $m \{ \cdots \}$ denotes the Lebesgue measure of $\{ \cdots \}$ and $\| \|_p$ the $L^p$ norm. As usual, the weak spectrum $\Sigma_w(A)$ of $A$ is defined to be $\Sigma - \rho_w(A)$. Obviously, $\Sigma_w(A) \subseteq \Sigma(A)$. That $\Sigma_w(A)$ can be a proper subset of $\Sigma(A)$ will be shown in §5.

3. On $\Sigma_w(T_{\sigma,c})$, $1 < p < \infty$. We first show that $\Sigma_w(T_{\sigma,c})$ is not empty.

THEOREM 3.1. If $\sigma(\xi)$ is not bounded away from a complex number $\lambda$ for all $\xi \in \mathbb{R}^n$, then $\lambda \in \Sigma_w(T_{\sigma,c})$.

PROOF. For simplicity, we suppose that $\lambda = 0$. Let $\{ \xi_k \}$ be a sequence of elements of $\mathbb{R}^n$ such that $\sigma(\xi_k) \to 0$ as $k \to \infty$. Let $\{ \varepsilon_k \}$ be a sequence of positive numbers. Let $\theta \in C_0^\infty(\mathbb{R}^n)$ be such that $\theta(\xi) = 0$ for $|\xi| > 1$ and $(2\pi)^{-n/2} \int_{\mathbb{R}^n} \theta(\xi) \, d\xi = 1$. Let
\(\psi \in \mathcal{S}\) be such that \(\hat{\psi} = \theta\). For \(k = 1, 2, \ldots\), define
\[
\varphi_k(x) = \varepsilon_k^{n/p} \psi(\varepsilon_k x) e^{i\varepsilon_k x}.
\]
If \(0 \in \rho_w(T_{ap})\), then there is a constant \(C > 0\) such that
\[
m\{ x \in \mathbb{R}^n : |\varphi_k(x)| > \alpha \} \leq \left\{ C \|T_{ap}\|/\alpha \right\}^p
\]
for all \(\alpha > 0\) and \(k = 1, 2, \ldots\). Choosing \(\alpha = \frac{1}{2} \varepsilon_k^{n/p}\) and using (3.1), inequality (3.2) becomes
\[
m\{ x \in \mathbb{R}^n : \left|\psi(\varepsilon_k x)\right| > \frac{1}{2} \} \leq \varepsilon_k^{-n} O\left(\|T_{ap}\|^p\right).
\]
Since \(\psi(0) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \theta(\xi) d\xi = 1\), it follows that there exists a \(\delta > 0\) such that \(|\psi(x)| > \frac{1}{2}\) whenever \(|x| \leq \delta\). Therefore
\[
m\{ x \in \mathbb{R}^n : \left|\psi(\varepsilon_k x)\right| > \frac{1}{2} \} \geq \pi \left(\delta/\varepsilon_k\right)^n
\]
for \(k = 1, 2, \ldots\). Hence, by (3.3),
\[
\pi \delta^n \leq O\left(\|T_{ap}\|^p\right)
\]
for \(k = 1, 2, \ldots\). But as has been shown in the proof of Theorem 3.1 in Wong [5], we can choose the \(\varepsilon_k\)’s going to zero so fast that \(\|T_{ap}\| \to 0\) as \(k \to \infty\). Thus (3.4) is impossible.

A useful consequence of Theorem 3.1 is

**Corollary 3.2.** \(\Sigma_w(T_{ap})\) contains the set \(\{ \sigma(x) : x \in \mathbb{R}^n \}\).

**4. Multipliers of weak type \((p, p)\), \(1 \leq p < \infty\).** Let \(m\) be a bounded measurable function on \(\mathbb{R}^n\). For any \(\varphi \in \mathcal{S}\), we define \(T_m\varphi\) by
\[
(T_m\varphi)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\xi \cdot x} m(\xi) \hat{\varphi}(\xi) d\xi.
\]
Suppose that there is a constant \(C > 0\) such that
\[
m\{ x \in \mathbb{R}^n : |(T_m\varphi)(x)| > \alpha \} \leq \left\{ C \|\varphi\|/\alpha \right\}^p
\]
for all \(\alpha > 0\) and \(\varphi \in \mathcal{S}\). Then we call \(T_m\) a multiplier of weak type \((p, p)\).

The connection between weak type multipliers and weak spectra is provided by

**Theorem 4.1.** A complex number \(\lambda\) is in the weak resolvent set \(\rho_w(T_{ap})\) of \(T_{ap}\) if and only if \(1/(\sigma(\xi) - \lambda)\) is a multiplier of weak type \((p, p)\).

**Proof.** We first prove necessity. Again for simplicity, let \(\lambda = 0\). By Theorem 3.1, \(\sigma(\xi)\) is bounded away from 0 for all \(\xi \in \mathbb{R}^n\). For any \(f \in \mathcal{S}\), define \(u\) by \(\hat{u}(\xi) = \hat{f}(\xi)/\sigma(\xi)\). Then \(u \in \mathcal{S}\) and \(T_0 u = f\). Since \(0 \in \rho_w(T_{ap})\), it follows that there is a constant \(C > 0\) such that
\[
m\{ x \in \mathbb{R}^n : |u(x)| > \alpha \} \leq \left\{ C \|f\|/\alpha \right\}^p
\]
for all \(\alpha > 0\) and \(f \in \mathcal{S}\). Hence \(1/\sigma(\xi)\) is a multiplier of weak type \((p, p)\). Conversely, if \(1/\sigma(\xi)\) is a multiplier of weak type \((p, p)\), then there is a constant \(C > 0\) such that
\[
m\{ x \in \mathbb{R}^n : |\varphi(x)| > \alpha \} \leq \left\{ C \|T_{ap}\|/\alpha \right\}^p
\]
for all $\alpha > 0$ and $\phi \in \mathcal{S}$. Since $\sigma(\xi)$ is bounded away from 0 for all $\xi \in \mathbb{R}^n$, it follows that $\mathcal{S} \subseteq R(T_{\phi})$. This proves that $R(T_{\phi})$ is dense in $L^p(\mathbb{R}^n)$. Consequently, $0 \in \rho_u(T_{\phi})$ if we can show that (4.1) is valid for all $\phi \in \mathcal{D}(T_{\phi})$.

**Lemma 4.2.** Inequality (4.1) is valid for all $\phi \in \mathcal{D}(T_{\phi})$.

**Proof.** For any $\phi \in \mathcal{D}(T_{\phi})$, let $\{\phi_k\}$ be a sequence of functions in $C_c^\infty(\mathbb{R}^n)$ such that $\phi_k \to \phi$ and $T_{\phi}\phi_k \to T_{\phi}\phi$ in $L^p(\mathbb{R}^n)$ as $k \to \infty$. Pick a subsequence of $\{\phi_k\}$, again denoted by $\{\phi_k\}$, such that $\phi_k \to \phi$ a.e. on $\mathbb{R}^n$. For any $\alpha > 0$, we set

$$E(\alpha) = \{x \in \mathbb{R}^n : |\phi(x)| > \alpha\}$$

and

$$E_k(\alpha) = \{x \in \mathbb{R}^n : |\phi_k(x)| > \alpha\}$$

for $k = 1, 2, \ldots$. Since $\phi \in L^p(\mathbb{R}^n)$, it follows that $m(E(\alpha)) < \infty$. So for any $\epsilon > 0$, by Egoroff’s Theorem, we can find a measurable set $A_\epsilon \subseteq \mathbb{R}^n$ such that $m(A_\epsilon) < \epsilon$ and $\phi_k(x) \to \phi(x)$ uniformly for all $x \in E(\alpha) - A_\epsilon$. Hence, there exists a positive integer $K$ such that $|\phi_k(x)| > \alpha$ whenever $k \geq K$. For such $k$’s, $E(\alpha) - A_\epsilon \subseteq E_k(\alpha)$, and using (4.1) and letting $k \to \infty$, we get

$$m(E(\alpha)) - \epsilon \leq C \|T_{\phi}\|/\alpha^p$$

for every $\alpha > 0$. Since $\epsilon$ is an arbitrary positive number, the proof is complete.

5. An example. We begin with an observation.

**Lemma 5.1.** For any $p$ such that $1 \leq p < \infty$, a sufficient condition for $\Sigma_w(T_{\phi}) = \Sigma(T_{\phi})$ is that $\Sigma(T_{\phi}) = \{\sigma(\xi) : \xi \in \mathbb{R}^n\}$.

Lemma 5.1 follows immediately from Corollary 3.2 and the fact that $\Sigma_w(T_{\phi}) \subseteq \Sigma(T_{\phi})$.

Let $\tau$ be the function defined by $\tau(\xi) = e^{-ik\xi}/(1 + |\xi|^c)$, where $0 < a < 1$ and $0 < c < na/2$. Then, defining $\sigma$ by $\sigma(\xi) = 1/\tau(\xi)$, it is clear that $\sigma \in S_{1/2,0}^\infty$. As has been proved in Wong [3, 4], $\Sigma(T_{\phi}) = \{\sigma(\xi) : \xi \in \mathbb{R}^n\}$ if $p$ is any number such that $1 < p < \infty$ and $|1/p - 1/2| < c/na$, and $\Sigma(T_{\phi}) = C$ if $|1/p - 1/2| > c/na$. The following result tells us that we know exactly what $\Sigma_w(T_{\phi})$ is if $1 < p < \infty$.

**Theorem 5.2.** $\Sigma_w(T_{\phi}) = \Sigma(T_{\phi})$, $1 < p < \infty$.

**Proof.** In view of Lemma 5.1, we need only consider $|1/p - 1/2| > c/na$. We first suppose that $1/p > 1/2 + c/na$. If $\lambda \in \Sigma(T_{\phi})$ and $\lambda \notin \Sigma_w(T_{\phi})$, then by Theorem 4.1, $1/(\sigma(\xi) - \lambda)$ is a multiplier of weak types $(2,2)$ and $(p,p)$. Hence, by the Marcinkiewicz interpolation theorem, $1/(\sigma(\xi) - \lambda)$ is an $L^q$ multiplier for any $q$ such that $1/2 + c/na < q < 1/p$. Thus, by Theorem 3.3 in Wong [3], $\lambda \in \rho(T_{\phi})$. But Theorem 3.1 in Wong [4] says that the spectrum of $T_{\phi}$ is either $C$ or $\{\sigma(\xi) : \xi \in \mathbb{R}^n\}$. Hence, $\Sigma(T_{\phi}) = \{\sigma(\xi) : \xi \in \mathbb{R}^n\}$. This is a contradiction. The proof for the case when $1/p < 1/2 - c/na$ is similar.

**Theorem 5.3.** $\Sigma_w(T_{\phi}) = \{\sigma(\xi) : \xi \in \mathbb{R}^n\}$. 

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PROOF. By Theorems 3.1 and 4.1, we need only show that if $\lambda \in \mathbb{C}$ is such that $\sigma(\xi) \neq \lambda$ for all $\xi \in \mathbb{R}^n$, $1/(\sigma(\xi) - \lambda)$ is a multiplier of weak type $(1,1)$. But an easy computation shows that $1/(\sigma(\xi) - \lambda) \in S^1_{1-a,0}$, and hence it follows from Theorem 2 in Fefferman [1] that $1/(\sigma(\xi) - \lambda)$ is indeed a multiplier of weak type $(1,1)$.

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