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STRUCTURE OF THE EFFICIENT POINT SET

DINH THE LUC

ABSTRACT. Let C be a nontrivial cone and X be a set in the n-dimensional Euclidean space. Denote by $\mathrm{E}(X|C)$ the set of all efficient points of X with respect to C. It will be proven that under some adequate assumptions $\mathrm{E}(X|C)$ is homeomorphic to a simplex while $n = 2$, and for $n > 2$ it is a contractible set. Furthermore, the set of all weak efficient points of X with respect to C is arcwise connected and its local contractibility is equivalent to being a retract of X. The results presented in this study cover all topological properties of the efficient point set which have been obtained by Peleg and Morozov for the case when C is the nonnegative orthant.

1. Introduction. The standard mathematical definition of Pareto optimum was first given by Debreu [**5**] in 1954 and soon became a fundamental concept in economic equilibrium theory, multicriteria optimization, game theory and other areas of mathematics. Smale [**11**], Schecter [**10**] and some other authors provided elegant descriptions of the differentiable structure of Pareto optima. The definition of Pareto optimum in the above-mentioned papers is based on differentials, therefore it is slightly different from the classical one. Concerning the classical Pareto optima, Peleg [**8**], Morozov [**7**] and Podinovskij [**9**] gave a deep study on the topological properties of the efficient point set. The topological features which have been pointed out in [**7–9**] are closely related to the fixed point property which is useful in proving the existence of equilibria for various market mechanisms, and for developing the iterative algorithm for determining the equilibria.

In relation to the Pareto optima in a more general setup such as that of Yu [**13**] or Corley [**4**], there are only a few works, and the authors investigate mainly the linear case where it is easy to derive Pareto optimum from the scalar representation (see, for example, [**2, 13**]). The purpose of our paper is to study the topological structure of the efficient point set with respect to a cone for some classes of convex sets in finite-dimensional Euclidean spaces. In fact, we shall prove in §3 that if a set is cone-convex and cone-closed, then its efficient point set with respect to the cone is contractible. This result may be improved in the 2-dimensional space, namely the efficient point set of a cone-convex, cone-compact set is homeomorphic to a simplex. In §4 we shall give a short presentation on the structure of the weak efficient point set. It will be proven that the weak efficient point set of a convex closed set is arcwise connected and is locally contractible if and only if it is a retract of the set. Earlier results of [**8, 7, 9** and **12**] concerning topological properties of the efficient point set can be derived from the results of the present paper as a particular case when the cone is the nonnegative orthant of the space.

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2. Preliminaries. Throughout this paper let R^n be the n-dimensional Euclidean space, let C be a cone and X a nonempty set in R^n.

DEFINITION 2.1. A point $x \in X$ is said to be an efficient point of X with respect to C if every $y \in X$ with $y - x \in C$ implies $y = x$.

DEFINITION 2.2. Assume that the interior $\operatorname{int} C$ of C is nonempty. It is said that $x \in X$ is a weak efficient point of X with respect to C if there is no $y \in X$ such that $y - x \in \operatorname{int} C$.

Let $\mathrm{E}(X|C)$ and $\mathrm{WE}(X|C)$ denote the set of all efficient points and the set of all weak efficient points of X with respect to C, respectively. In the case when C is the nonnegative orthant R^n_+ of R^n, $\mathrm{E}(X|C)$ is called the Pareto optimal set of X and let it be denoted by $\mathrm{PO}(X)$, and similarly let $\mathrm{WE}(X|C)$ be denoted by $\mathrm{WPO}(X)$.

In economic theory the definition of Pareto optima is often presented in the following manner. It is given a set of alternatives available to n agents and each agent has a utility function over the set of alternatives. An alternative x is called a Pareto optimum if there is no other alternative y such that the value of every utility function at y is not smaller and at least one of them is greater than the one at x. This definition of a Pareto optimum becomes the one we defined when transformed to the utility space, i.e. the space of the values of utility functions.

DEFINITION 2.3. It is said that X is C-convex, C-closed if the set $X - C$ is convex, closed, respectively. If there is a bounded set $Y \subset R^n$ such that $X - C \subset Y - C$, then X is called C-bounded. If X is C-closed and C-bounded, then it is called C-compact.

DEFINITION 2.4. X is said to be C-strictly convex if it is a C-convex set with nonempty interior, and for each $x, y \in X$, $x \neq y$, the set $(x + y)/2 + C$ has a nonempty intersection with $\operatorname{int} X$.

DEFINITION 2.5. X is said to be C-quasistrictly convex provided it is a C-convex set and for each $x \in X \setminus \mathrm{E}(X|C)$ the set $x + C$ has a nonempty intersection with the relative interior $\operatorname{ri}(X - C)$ of $X - C$. If $C = R^n_+$, then we say that X is quasistrictly convex.

REMARK 2.1. Every C-strictly convex set is C-quasistrictly convex; however, the inverse is not true even when X is convex and its interior is nonempty. For instance, the triangle with vertices $(0,0)$, $(0,1)$ and $(1,0)$ in the space R^2 is quasistrictly convex but not R^2_+-strictly convex.

We recall that C is pointed if $C \cap (-C) \subset \{0\}$, and C^* denotes the nonnegative polar cone of C, i.e. $C^* = \{p \in R^n\colon \langle p, x\rangle \geqslant 0 \text{ for each } x \in C\}$. The following assumption will be imposed on C and X, although many of our results which will be proven remain valid without this assumption:

Assumption 2.1. C is convex, closed and pointed; X is C-closed and C-convex.

The following lemmas are either easy to prove or they may be found in [**4** and **13**]. Therefore, the proofs are omitted.

LEMMA 2.1. *$\mathrm{E}(X|C)$ is nonempty if and only if $X - C$ and C have no common direction of recession.*

LEMMA 2.2. $\mathrm{E}(X|C) = \mathrm{E}(X - C|C)$.

LEMMA 2.3. *Let T be a nondegenerated affine transformation of R^n. Then* $T(\mathrm{E}(X|C)) = \mathrm{E}(TX|TC)$.

LEMMA 2.4. *Let $\mathrm{E}_p(X)$ be the set of all points of X which solve the optimization problem* $\max\langle p, x\rangle$, *subject to $x \in X$, where p is an arbitrary vector of R^n. Then the following relations hold*:

$$\bigcup_{\substack{p \in \operatorname{int} C^* \\ \|p\|=1}} \mathrm{E}_p(X) \subseteq \mathrm{E}(X|C) \subseteq \bigcup_{\substack{p \in C^* \\ \|p\|=1}} \mathrm{E}_p(X).$$

LEMMA 2.5. *If, in addition to Assumption* 2.1, *we suppose* $\operatorname{int} C \neq \varnothing$, *then*
(i) $\bigcup_{p \in C^*, \|p\|=1} \mathrm{E}_p(X) = \mathrm{WE}(X|C)$,
(ii) $\mathrm{WE}(X|C) = \mathrm{E}(X|C)$ *if and only if X is C-quasistrictly convex.*

PROOF. Observe that $x \in \mathrm{WE}(X|C)$ if and only if $x \in X$ and the convex sets $x + \operatorname{int} C$ and $X - C$ are disjoint. Now, the first statement is a simple consequence of the separation theorem (see [**1**]). The second statement can be verified directly by using Definitions 2.2 and 2.5. □

In order to formulate further results more definitions are needed from [**1** and **3**]. For the sake of convenience they will be mentioned next.

Let F be a set-valued map from a set $A \subset R^n$ into R^n. F is said to be upper semicontinuous at $x_0 \in A$ if for any open set U which contains $F(x_0)$ there exists a neighborhood V of x_0 such that $x \in A \cap V$ implies $F(x) \subset U$. F is said to be lower semicontinuous at x_0 if for any open set U which meets $F(x_0)$ there exists a neighborhood V of x_0 such that U meets $F(x)$ for every $x \in A \cap V$. If F is upper and lower semicontinuous at $x \in A$, we say F is continuous at x. F is said to be Hausdorff continuous on A provided it is continuous at every point of A and $F(x)$ is compact for each $x \in A$.

Furthermore, a set A is said to be contractible if there exist a continuous map $H(x, t)\colon A \times [0,1] \to A$ and a point $x' \in A$ such that

$$(1) \qquad H(x,0) = x' \quad \text{and} \quad H(x,1) = x \quad \text{for each } x \in A.$$

A subset $B \subset A$ is said to be a retract of A if there exists a continuous map h: $A \to B$ such that

$$(2) \qquad h(x) = x \quad \text{for each } x \in B.$$

Finally, a set A is said to be locally contractible provided any neighborhood V of any point $x \in A$ contains a neighborhood $U \ni x$ such that $U \cap A$ is contractible.

3. Structure of the efficient point set. In this section, in addition to Assumption 2.1, we suppose that $\mathrm{E}(X|C)$ is nonempty. Moreover, by passing to the space of smaller dimension if necessary it may be assumed that $\operatorname{int}(X - C)$ is nonempty. In further discussions we shall write $\mathrm{E}(X)$ instead of $\mathrm{E}(X|C)$ if it does not lead to confusion.

Let G be a set-valued map defined on $X - C$ as follows:

$$G(x) = (x + C) \cap (X - C) \quad \text{for } x \in X - C.$$

It is clear that $G(x)$ is nonempty, convex and compact for each $x \in X - C$.

LEMMA 3.1. *Let* $x \in \operatorname{int}(X - C)$ *and* $y \in G(x)$. *For every neighborhood* U *of* y *there exists a neighborhood* V *of* x *in* $\operatorname{int}(X - C)$ *such that* $V \subset [U \cap (X - C)] - C$.

PROOF. Suppose to the contrary that for some neighborhood U of y there exists a sequence $\{x_k\}$, $x_k \in \operatorname{int}(X - C)$, $\lim x_k = x$ such that

$$x_k \notin [U \cap (X - C)] - C. \tag{3}$$

Take an open ball V with centre x and radius ε such that $V \subset \operatorname{int}(X - C)$. By the convexity of $X - C$, the convex hull of V and y lies in $X - C$. Moreover, for each t from the open interval between 0 and 1 the ball V_t with centre $y_t = ty + (1 - t)x$ and radius $(1 - t)\varepsilon$ belongs to $X - C$ and $V_t - C$ contains x as an interior point. Now, for t being near to 1, V_t lies in $U \cap (X - C)$ and this contradicts (3). Thus the proof is complete. □

LEMMA 3.2. $G(x)$ *is Hausdorff continuous at every point of* $\operatorname{int}(X - C) \cup \mathrm{E}(X)$.

PROOF. Suppose first that $x_0 \in \mathrm{E}(X)$. We have $G(x_0) = \{x_0\}$. Let U be an open set containing x_0. If there were a sequence $\{x_k\}$ with $\lim x_k = x_0$, $x_k \in X - C$, and a sequence $\{y_k\}$ with $y_k \in G(x_k)$ and $y_k \notin U$; then any cluster point y_0 of $\{y_k\}$ would be different from x_0, and moreover, $y_0 - x_0 \in C$, contradicting $x_0 \in \mathrm{E}(X)$. Thus, the upper semicontinuity of $G(x)$ at x_0 is proven, and its lower semicontinuity can also be verified similarly by using the fact that any open set meeting $G(x_0)$ contains $G(x_0)$. Now, let $x_0 \in \operatorname{int}(X - C)$. for the lower semicontinuity we suppose to the contrary that there are an open set U meeting $G(x_0)$ and a sequence $\{x_k\}$ with $x_k \in \operatorname{int}(X - C)$, $\lim x_k = x_0$ such that

$$G(x_k) \cap U = \varnothing. \tag{4}$$

Take $y_0 \in G(x_0) \cap U$ and apply Lemma 3.1 to get a sequence $\{y_k\}$, $y_k \in (x_k + C) \cap (X - C) = G(x_k)$ with $\lim y_k = y_0$, contradicting (4). The upper semicontinuity follows immediately from the closedness of $X - C$ and that of $G(x)$. Thus the lemma is proven. □

REMARK 3.1. It can be easily seen that $G(x)$ is upper semicontinuous on $X - C$, however, it is not necessary for G to be lower continuous on $X - C$. It can also be proven that G is continuous on $X - C$ if X is C-strictly convex.

COROLLARY 3.1. *If* X *is* C-*quasistrictly convex, then* $G(x)$ *is Hausdorff continuous on* X.

PROOF. Under the assumption of the corollary we have $X \subset \operatorname{int}(X - C) \cup \mathrm{E}(X)$. Then apply Lemma 3.2 to get the assertion.

LEMMA 3.3. *If* X *is* C-*quasistrictly convex, then* $\mathrm{E}(X)$ *is closed.*

PROOF. Let $\{y_k\}$ be a sequence of efficient points of X with $\lim y_k = y$ being in $X - C$ by the C-closedness of X. Suppose to the contrary that $y \notin \mathrm{E}(X)$. Take a point z with $z \neq y$, $z \in (y + C) \cap \operatorname{int}(X - C)$ and a neighborhood V of 0 such that $z + V \subset X - C$. For large values of k, $y_k = y + v_k$ where $v_k \in V$. Hence $z + v_k \in y_k + C$ with $z + v_k \in X - C$. Thus $y_k \notin \mathrm{E}(X)$, and this contradiction completes the proof. □

REMARK 3.2. In general, without the assumption made in the previous lemma, $\mathrm{E}(X)$ may not be closed in the case of $X \subset R^n$, $n > 2$. For $n = 2$ the structure of $\mathrm{E}(X)$ is very simple, as shown by the following result.

THEOREM 3.1. *Assume that X is C-compact and $X \subset R^2$. Then* $\mathrm{E}(X)$ *is homeomorphic to a simplex.*

PROOF. Consider first the case when $\dim C = 2$. By Lemma 2.3 we may assume $C = R^2_+$ and $X \subset \operatorname{int} R^2_+$. For the closedness of $\mathrm{PO}(X)$, let $\{x_k\}$ be a sequence of points of $\mathrm{PO}(X)$ with $\lim x_k = x \in X - R^2_+$. If x was not a Pareto optimal point of X, then there would be a point $x \in X$ with $y > x$, say $y^1 > x^1$, $y^2 \geqslant x^2$ where $x = (x^1, x^2)$, $y = (y^1, y^2)$. For large values of k we have $x_k^2 > y^2$ since $x_k \in \mathrm{PO}(X)$. Consider the segment $[x_{k_0}, y]$ for a fixed, sufficiently large k_0. Obviously this segment lies in $X - C$ and the set $[x_{k_0}, y] - \operatorname{int} C$ contains x. When x_k converges to x we arrive at $x_k \in \mathrm{PO}(X)$, a contradiction, and consequently $\mathrm{PO}(X)$ is closed. Now, we show that the cone generated by $\mathrm{PO}(X)$ is convex. Suppose that a and b are two different points of $\mathrm{PO}(X)$ and $c = \lambda a + (1 - \lambda)b$ for some $\lambda \in [0, 1]$. Consider the optimization problem

$$\max t, \quad \text{subject to } t \geqslant 0 \text{ and } tc \in X - C.$$

It is clear that this problem has an optimal solution, say t'. We shall next prove that $t'c \in \mathrm{PO}(X)$. Indeed, if $t'c$ was not in $\mathrm{PO}(X)$, then there would be a point $x \in X$ with $x \in t'c + C$, $x \neq t'c$. Observe that the triangle with vertices a, b and x is contained in $X - C$ and it contains $t'c$ as an interior point. This relation contradicts the choice of t'. Hence the convexity of the cone is established. Furthermore, for each $c \in \mathrm{PO}(X)$ the ray starting from 0 through c meets $\mathrm{PO}(X)$ at the only point c. This fact shows that $\mathrm{PO}(X)$ is homeomorphic to the simplex which generates the above convex, closed cone.

For the case $\dim C = 1$, as noted in the beginning of the proof, we may assume $X \subset \operatorname{int} R^2_+$ and C is one of the coordinate axes, say the first one. Let x_0 and y_0 solve the following problems:

$$\min x^2, \quad \text{subject to } x = (x^1, x^2) \in X - C, x^1 = 0,$$

and

$$\max x^2, \quad \text{subject to } x = (x^1, x^2) \in X - C, x^1 = 0,$$

respectively. These solutions exist and are unique by the C-compactness and C-convexity of X. We shall next verify that $\mathrm{E}(X)$ is homeomorphic to the segment $[x_0, y_0]$. For, construct two maps f: $[x_0, y_0] \to \mathrm{E}(X)$ and f': $\mathrm{E}(X) \to [x_0, y_0]$ as follows: for $x = (0, x^2) \in [x_0, y_0]$, $f(x) = y$ where $y = (y^1, y^2)$ with $y^1 = \max\{t > 0\colon (t, y^2) \in X - C\}$, $y^2 = x^2$, for $y = (y^1, y^2) \in \mathrm{E}(X)$, $f'(y) = (0, y^2)$. It is clear that f and f' are correctly defined and continuous. These maps provide a homeomorphism between $[x_0, y_0]$ and $\mathrm{E}(X)$. The proof is complete. □

REMARK 3.3. For an arbitrary strictly convex compact set X in R^n with $n > 2$, it is not necessary for $\mathrm{PO}(X)$ to be homeomorphic to a simplex. In the following example, we shall give a quasistrictly convex compact set in R^3 with the Pareto optimal set not being homeomorphic to a simplex. It is easy to modify this example for the strictly convex case.

EXAMPLE 3.1. Let X be the polyhedron in R^3 with the following five vertices: $(3,2,0)$, $(2,3,0)$, $(4,0,0)$, $(0,4,0)$ and $(2.6,2.6,3)$. This polyhedron is a quasistrictly convex set. Its Pareto optimal set consists of two triangles, one with vertices $(4,0,0)$, $(3,2,0)$, $(2.6,2.6,3)$ and another with vertices $(0,4,0)$, $(2,3,0)$, $(2.6,2.6,3)$. These triangles have only one common point $(2.6,2.6,3)$ and their union cannot be homeomorphic to a simplex.

THEOREM 3.2. $\mathrm{E}(X)$ *is contractible.*

PROOF. Let R^k be the smallest subspace of R^n which contains C. Let C' denote the nonnegative polar cone of C in R^k. Since C is pointed, there exists $t' \in \operatorname{ri} C'$ such that $\langle t', e\rangle = 1$ for some $e \in \operatorname{ri} C$. Construct a map $F(x)$: $X - C \to R^n$ as follows:

$$F(x) = x + (\alpha - \langle t', x\rangle)e/\beta,$$

where α and β are constants which are defined next. Set $\alpha = \max\{\langle t', x\rangle : x \in X - C\}$. Since $t' \in \operatorname{ri} C' \subseteq \operatorname{int} C^*$ and $\mathrm{E}(X) \neq \varnothing$ the value of α is finite. Also, since $t' \in \operatorname{int} C^*$ there exists a positive ε such that $t' \in t + C$ for each $t \in C^*$ with $\|t\| \leqslant \varepsilon$. Moreover, as $e \in \operatorname{ri} C$, $\beta = \varepsilon \cdot \min\{\langle t, e\rangle : t \in C', \|t\| = 1\}$ will be positive. Hence $F(x)$ is correctly defined and continuous on $X - C$. We shall prove that

$$F(x) \in y + C \quad \text{for each } y \in G(x). \tag{5}$$

Indeed,

$$F(x) \in x + \langle t', y - x\rangle \cdot e/\beta + C \quad \text{for each } y \in G(x). \tag{6}$$

Moreover, from the definition of β and from the fact that $y - x \in C$ it follows that

$$\langle t' \cdot \langle e, t\rangle/\beta - t,\ y - x\rangle \geqslant 0 \quad \text{for each } t \in C' \text{ with } \|t\| = 1.$$

The latter relation is equivalent to

$$\langle\langle t', y - x\rangle \cdot e/\beta - (y - x), t\rangle \geqslant 0 \quad \text{for each } t \in C',\ \|t\| = 1. \tag{7}$$

Since $\langle t', y - x\rangle \cdot e/\beta - (y - x) \in R^k$, (7) implies

$$\langle t', y - x\rangle \cdot e/\beta - (y - x) \in C'', \tag{8}$$

where C'' is the nonnegative polar cone of C' in R^k. By the closedness of C we have $C'' = C$. Now, to get (5) it suffices to combine (8) with (6). Furthermore, let f be a map from $X - C$ into $\mathrm{E}(X)$ which is defined in the following way: $f(x) = y$ where y solves the optimization problem

$$\min\|z - F(x)\|, \quad \text{subject to } z \in G(x).$$

In virtue of (5) and the fact that $G(x)$ is convex compact, the above map is correctly defined. Moreover, if $x \in \mathrm{E}(X)$ then $G(x) = \{x\}$ and consequently $f(x) = x$. Furthermore, by Lemma 3.2, $G(x)$ is continuous at every point of $\operatorname{int}(X - C) \cup \mathrm{E}(X)$ and so is $f(x)$. The map $H(x, t)$ from $\mathrm{E}(X) \times [0,1]$ into $\mathrm{E}(X)$ may be defined as follows:

$$H(x, t) = f(tx + (1 - t)a),$$

where a is some fixed point from $\operatorname{int}(X - C)$. It is clear that $H(x, t)$ is continuous and satisfies (1). In this way, $\mathrm{E}(X)$ is contractible and the theorem is proven. □

THEOREM 3.3. $\mathrm{E}(X)$ *is closed if and only if it is a retract of* $X - C$.

PROOF. Since $X - C$ is closed, therefore any retract of this set is closed. Now suppose that $\mathrm{E}(X)$ is closed. Denote by $d(x, \mathrm{E}(X))$ the distance from x to $\mathrm{E}(X)$ and let $t(x) = d(x, \mathrm{E}(X))/(1 + d(x, \mathrm{E}(X)))$. Define $h(x)\colon X - C \to \mathrm{E}(X)$ by

$$h(x) = f((1 - t(x))x + t(x)a),$$

where a is a fixed point from $\operatorname{int}(X - C)$ and f is the function defined in the proof of Theorem 3.2. For any $x \in X - C$ we have

$$(1 - t(x))x + t(x)a \in \operatorname{int}(X - C) \cup \mathrm{E}(X).$$

Therefore, $h(x)$ is continuous on $X - C$. Moreover, if $x \in \mathrm{E}(X)$, then $h(x) = f(x) = x$. Consequently, (2) holds and the theorem is proven. □

COROLLARY 3.2. *Assume that X is C-quasistrictly convex. Then* $\mathrm{E}(X)$ *is a retract of the convex hull of X.*

PROOF. By Lemma 3.3, $\mathrm{E}(X)$ is closed. The restriction of the map $h(x)$ being defined in the previous proof on the convex hull of X will satisfy (2) and this observation completes the proof. □

4. Structure of the weak efficient point set. It is assumed in this section that C is a pointed, convex, closed cone with nonempty interior and X is a closed, convex set.

THEOREM 4.1. *If* $\mathrm{E}(X)$ *is nonempty, then* $\mathrm{WE}(X)$ *is arcwise connected.*

PROOF. Let x and $y \in \mathrm{WE}(X)$. It is easy to see that $(x + C) \cap X$ and $(y + C) \cap X$ belong to $\mathrm{WE}(X)$ and have nonempty intersections with $\mathrm{E}(X)$. Let x' and y' be two points from these intersections. In virtue of Theorem 3.2, x' and y' can be connected by some arc in $\mathrm{E}(X)$. This arc and the segments $[x, x']$, $[y, y']$ will connect x with y in $\mathrm{WE}(X)$. Thus, the proof is complete. □

REMARK 4.1. If $\mathrm{E}(X) = \varnothing$, then $\mathrm{WE}(X)$ may be disconnected. For example, in the 3-dimensional space R^3, let X be the convex hull of the following points: $(0,0,0)$, $(0,1,0)$, $(1,0,0)$, $(0,0,k)$, $(0,1,k)$, $(1,0,k)$ and $(1 - 1/2^k, 1 - 1/2^k, k)$, $k = 1, 2, \ldots$. Then $\mathrm{PO}(X) = \varnothing$ and

$$\mathrm{WPO}(X) = \{(1,0,z)\colon z \geqslant 0\} \cup \{(0,1,z)\colon z \geqslant 0\}.$$

THEOREM 4.2. *Assume that* $\mathrm{E}(X) \neq \varnothing$. $\mathrm{WE}(X)$ *is locally contractible if and only if it is a retract of X.*

PROOF. Since X is a convex closed set, it is locally contractible, and so is any retract of X. Now, suppose that $\mathrm{WE}(X)$ is locally contractible. By Dugundji's theorem [**6**, Theorem 15.1] it follows that $\mathrm{WE}(X)$ is a retract of some neighborhood V of $\mathrm{WE}(X)$ in X, i.e. there exists a continuous map $h_1(x)$ from V into $\mathrm{WE}(X)$ satisfying (2). Furthermore, for every integer $k > 0$ there exists a number $s(k) > 0$ such that

$$h_2(x) = (x + sd(x, W)f(x))/(1 + sd(x, W))$$

lies in V for each $x \in X$ with $\|x\| \leqslant k$, $s \geqslant s(k)$, where f is the map constructed in the proof of Theorem 3.2, and $d(x, W)$ is the distance from x to $\mathrm{WE}(X)$. Without loss of generality we may assume that $s(k+1) \geqslant s(k)$ for each k.

Let

$$s(t) = [s(k+2) - s(k+1)]t + (k+1)s(k+1) - ks(k+2)$$

for $t \in [k, k+1]$, $k = 1, 2, \ldots$, and let

$$h_3(x) = \big(x + s(\|x\|)d(x, W)f(x)\big)/\big(1 + s(\|x\|)d(x, W)\big).$$

It is obvious that h_3 is a continuous map on X. Moreover, $h_3(x) \in V$ for each $x \in X$ and $h_3(x) = x$ if $x \in \mathrm{WE}(X)$. Now the composition $h_1 h_3(x)$ will be a continuous map from X into $\mathrm{WE}(X)$ which satisfies (2). Thus $\mathrm{WE}(X)$ is a retract of X and the proof is complete. □

5. Conclusion. In the present paper we have only concerned ourselves with the utility space, i.e. the space where utility functions take their values. The reader will be asked to derive the earlier results obtained by the authors of [**7**–**9** and **12**] from ours, except for those of [**12**] concerning the space of the alternatives, which we shall point out in a forthcoming paper.

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Computer and Automation Institute, Hungarian Academy of Sciences, Kende U. 13-17, Budapest H-1502, Hungary